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# Martin Boundary and Elliptic Curves

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**Abstract.** Martin boundary is found for two-dimensional transient random walks on a plane lattice with different jumps in a finite number of other points, a half-plane and a quarter-plane. The random walks are homogeneous outside the boundary and possibly in a finite number of other points. The approach is based on the analysis of the elliptic curve defined by the jump generating function. In most cases the Martin boundary is proved to be homeomorphic to some subset of "real" points of this curve. In other cases the minimal Martin boundary consists of one or two points.

Keywords: random walk, Martin boundary, Riemann surface, algebraic function, saddle-point method

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## Introduction

There exist only a few non-trivial examples where it is possible to find the Martin boundary for transient Markov chains. One such example is the homogeneous random walk on a lattice, see Ney and Spitzer [12]. Due to complete homogeneity they succeeded to find it, using only rather elementary analytic tools. They used a change of measure quite similar to the one used in large deviation problems. However, in most cases probabilistic methods hardly help in this kind of problems. For example, if we change the jumps in only one point, the method of [12] does not work. In this paper we consider some examples with piecewise linear homogeneities: the plane lattice with possibly different jumps in a finite number of points, the half-plane, the quarter-plane.

Why are these problems interesting?

Note first that random walks in unbounded domains with non-smooth boundaries appear naturally in many applied fields, for example in queueing networks.

The simplest non-trivial example is a quarter plane, and for a long time it was a laboratory for development of probabilistic and analytic methods in this field. There are some problems that can be solved by probabilistic methods (mainly martingales) such as classification problems, large deviations (see [5]), intrinsic convergence rates (see [7]), Poisson boundary (see [6]). But for the Martin boundary probabilistic methods hardly could be applied and so we use analytic methods, in particular complex analysis on the algebraic (elliptic) curve that is defined by the generating function of the jumps inside the quarter-plane. Note however, that we do not use analytic methods in full extent: we do not need explicit solution for functional equations, but only analytic continuation properties. We show that the Martin boundary is related to real points of this curve. This could be vaguely predicted, because real points also play a main role in the study of large deviations. However, the connection between Martin boundary and large deviation paths is still obscure and we hope that this work will give rise to some hypotheses in this direction. One could speculate that the existence of such connections is quite plausible: in homogeneous cases the paths to the Martin boundary and the large deviation paths are linear, the change of measure for both problems is the same in some simple situations; the intrinsic convergence rate for an ergodic random walk is also related to boundaries and to large deviations, etc.

The structure of the paper is the following. In Section 1 we give all necessary definitions concerning Martin boundary, so as to render this paper self-contained and we define the process. In Section 2 main results are formulated. This section is split into seven subsections. In Subsection 2.1 we give the indispensable information on the elliptic curve, that is used to present the results on the Martin boundary. Subsections 2.2–2.7 are devoted to the Martin boundary for the transient random walk in

- the plane $\mathbf{Z}^2$,
- the half-plane $\mathbf{Z} \times \mathbf{Z}_+$ with escape to infinity along the internal part,
- the half-plane $\mathbf{Z} \times \mathbf{Z}_+$ with escape to infinity along the axis,
- the quarter-plane $\mathbf{Z}_+^2$ with escape to infinity along the internal part,
- the quarter-plane $\mathbf{Z}_+^2$ with escape to infinity along one axis,
- the quarter-plane $\mathbf{Z}_+^2$ with escape to infinity along two axes

respectively.

Section 3 is divided into seven subsections and contains the proofs of the results claimed. In Subsection 3.1 we describe the structure of these proofs.

The following directions of future research may be pursued.

1. For more general jumps in dimension two, one could apply analytic methods as well and obtain the localisation of Martin boundary on the corresponding algebraic curve.

2. Extend the results of this paper for exit and entrance boundaries for the recurrent case.

The following question may be also investigated: do some general connections between large deviations and Martin boundary exist as is the case in some examples?

## 1. Main definitions and the process

A discrete time homogeneous Markov chain $\mathcal{L}$ on a denumerable state space $X$ is defined by the stochastic matrix $P = \big(p_{\alpha\beta}\big)$, $\alpha, \beta \in X$, and an initial distribution $\theta$. Let the matrix elements of $P^n$ be $p^n_{\alpha\beta}$. We denote the probability measure on the path space $X^\infty = \{(\alpha_n)_{n=o}^\infty, \alpha_n \in X\}$ by $\mathsf{P}_\theta$. We will denote the probability measure in $X^\infty$ by $\mathsf{P}_\alpha$, if the initial state is $\alpha$. We will only study *irreducible aperiodic* Markov chains (see [3] for definitions). Let us introduce the Green function $\pi^\alpha_\beta$ as the mean number of visits to $\beta$ starting from $\alpha$:

$$\pi^\alpha_\beta = \sum_{n=0}^\infty p^n_{\alpha\beta} = \mathsf{E}_\alpha \sum_{n=0}^\infty 1\{X_n = \beta\}.$$

**Definition 1.1.** An irreducible aperiodic Markov chain is called *transient* if $\pi^\alpha_\beta < \infty$ for some $\alpha, \beta \in X$. A Markov chain that is not transient, is called *recurrent*.

For an irreducible aperiodic transient chain, it follows that $\pi^\alpha_\beta < \infty$ for all ordered pairs $\alpha, \beta \in X$.

### 1.1. Martin boundary

In this subsection we introduce the Martin boundary and we formulate related basic results. All of them are well-known, see [2] or [13].

Let us fix a probability measure $\eta(\alpha)$ on $X$, such that $\sum_{\alpha\in X}\eta(\alpha)\pi^\alpha_\beta > 0$ for all $\beta \in X$. This will be the so-called "reference" measure. The construction of the Martin boundary depends heavily on the choice of this measure. It may happen that the boundaries obtained for two different reference measures are not homeomorphic.

The Martin kernel for the transient chain $\mathcal{L}$ is defined as

$$\mathsf{k}_\beta(\alpha) = \frac{\pi^\alpha_\beta}{\sum\limits_{\gamma\in X}\eta(\gamma)\pi^\gamma_\beta}.$$

Note that $\pi^\alpha_\beta = p(\alpha,\beta)\pi^\beta_\beta$, where $p(\alpha,\beta)$ is the probability to reach state $\beta$ from $\alpha$. In view of the inequality $p(\gamma,\beta) \geq p(\gamma,\alpha)p(\alpha,\beta)$, we have

$$\mathsf{k}_\beta(\alpha) = \frac{p(\alpha,\beta)}{\sum\limits_{\gamma\in X}\eta(\gamma)p(\gamma,\beta)} \leq \frac{1}{\sum\limits_{\gamma\in X}\eta(\gamma)p(\gamma,\alpha)} \stackrel{\text{def}}{=} \frac{1}{a(\alpha)}.$$

Next, we "enumerate" all states of the chain in arbitrary order. Let $N(\alpha) \in \mathbf{N}$ be the number of state $\alpha$. Define the distance $\rho$ in the state space $X$ by

$$\rho(\beta,\gamma) = |2^{-N(\beta)} - 2^{-N(\gamma)}| + \sum_{\alpha\in X} |\,\mathsf{k}_\beta(\alpha) - \mathsf{k}_\gamma(\alpha)|a(\alpha)2^{-N(\alpha)}. \tag{1.1}$$

For all $\beta, \gamma \in X$, we have $\rho(\beta,\gamma) \le 3$.

**Definition 1.2.** The compactification $X^*$ of the state space $X$ with respect to the distance (1.1) is called the *Martin compactification*; $\partial X = X^*/X$ is called the *Martin boundary.*

The choice of the distance (1.1) is not compulsory, provided that a sequence of states $\beta_n$ is a Cauchy sequence if and only if

1) the functions $\mathsf{k}_{\beta_n}(\alpha)$ converge in each point $\alpha$;
2) $N(\beta_n) \to \infty$ or $\beta_n$ are constant for $n \ge n_0$.

We will call the topology in the space $X^*$ induced by the distance $\rho(\alpha,\beta)$, the $M^+$ topology. The following theorem holds.

**Theorem 1.1.** *Let an arbitrary $\alpha \in X$ be an initial state of the chain. Then $\mathsf{P}_\alpha$-almost all sequences of states $\beta_n$ have limits in the topology $M^+$*

$$\lim_{n\to\infty} \beta_n = \beta_\infty \in \partial X.$$

If we define the measure $\mu_1$ on the Borel subsets $\Gamma$ of $X^*$ by

$$\mu_1(\Gamma) = \mathsf{P}_\eta\{\beta_\infty \in \Gamma\},$$

then for all $\alpha \in X$, $\mu_1(\alpha) = 0$ and the following theorem holds.

**Theorem 1.2.** *For all Borel function $f$ on $X^*$*

$$\mathsf{E}_\alpha\, f(\beta_\infty) = \int_{\partial X} \mathsf{k}_\beta(\alpha) f(\beta)\, \mu_1(d\beta).$$

One of the main purposes of Martin boundary theory is to give an integral representation of superharmonic and harmonic functions.

**Definition 1.3.** The function $h(\alpha)$ is called superharmonic if

$$Ph(\alpha) = \sum_{\beta\in X} p_{\alpha\beta} h(\beta) \le h(\alpha).$$

If the equality takes place, the function $h(\alpha)$ is called harmonic.

We consider only *non-negative* superharmonic and harmonic functions.

**Theorem 1.3.** *For any $\eta$-integrable superharmonic function $h(\alpha)$ there exists a finite measure $\mu_h$ on $X^*$ called a spectral measure of $h(\alpha)$, such that*

$$h(\alpha) = \int_{X^*} \mathsf{k}_\beta(\alpha)\mu_h(d\beta). \tag{1.2}$$

*Moreover, for all $\beta \in X$*

$$\mu_h(\beta) = (h(\beta) - Ph(\beta)) \sum_{\gamma \in X} \eta(\gamma)\pi_\beta^\gamma. \tag{1.3}$$

The representation (1.2) is called the Martin representation. In view of (1.3) it can be written in the following form

$$h(\alpha) = \sum_{\beta \in X} (h(\beta) - Ph(\beta))\pi_\beta^\alpha + \int_{\partial X} \mathsf{k}_\beta(\alpha)\,\mu_h(d\beta). \tag{1.4}$$

The spectral measure $\mu_h$ in (1.4) is generally not unique. In order to get uniqueness, we will introduce the *minimal* Martin boundary. Note that for fixed $\beta$, the function $\pi_\beta^\alpha$ is superharmonic as a function of $\alpha$ and so is $\mathsf{k}_\beta(\alpha)$. For all fixed $\alpha \in X$, $\mathsf{k}_\beta(\alpha)$ can be continuously extended to $X^*$. Indeed, if $\beta_n \to \beta_\infty$, then $\mathsf{k}_{\beta_n}(\alpha) \to \mathsf{k}_{\beta_\infty}(\alpha)$ for all $\alpha$. Thus for all $\beta \in \partial X$, the function $\mathsf{k}_\beta(\alpha)$ is superharmonic too.

**Definition 1.4.** A non-zero superharmonic function $h$ is said to be *minimal* if the equality $h = h_1 + h_2$ implies that $h_1 = c_1 h$, $h_2 = c_2 h$, where $c_1, c_2$ are constants and $h_1$, $h_2$ are superharmonic.

It follows that a harmonic function $h$ is minimal if for any other harmonic function $h_1 \leq h$ we have $h_1 = ch$.

**Definition 1.5.** The set

$$B = \{\beta \in \partial X : \ \ \mathsf{k}_\beta(\alpha) \text{ is minimal}\}$$

is called the *minimal Martin boundary.*

**Lemma 1.1.** *The set of minimal harmonic and superharmonic functions is*

$$\{c \cdot k_\beta(\alpha) : \ \beta \in B \cup X\}.$$

**Theorem 1.4.** *The set $B$ is a Borel subset of $\partial X$. For any $\beta \in B$, $\mathsf{k}_\beta(\alpha)$ is a harmonic function.*

**Theorem 1.5.** *Every $\eta$-integrable superharmonic function $h(\alpha)$ has the unique representation*

$$h(\alpha) = \int_{X \cup B} \mathsf{k}_\beta(\alpha)\,\mu(d\beta), \tag{1.5}$$

*where $\mu$ is a measure on the Borel sets of $X \cup B$. The measure $\mu$ is finite.*

*For all finite measures $\mu$ on $X \cup B$ the right-hand side of (1.5) defines a superharmonic $\eta$-integrable function.*

*This function is harmonic if and only if $\mu(X) = 0$.*

So, if the function $h(\alpha)$ can be represented in the form (1.5), then $\mu$ coincides with the spectral measure $\mu_h$ and the representation (1.5) coincides with the Martin representation (1.2) or (1.4). In particular, the measure $\mu_1(\Gamma) = \mathsf{P}_\eta\{\beta_\infty \in \Gamma\}$ is the spectral measure of $h(\alpha) = 1$. Moreover, $\mu_h(X/B) = 0$ for all superharmonic functions $h(\alpha)$ and $\mu_{\mathsf{k}_\beta} = \delta_\beta$ for all $\beta \in X \cup B$.

**Theorem 1.6.** *Let $\varphi(\beta_\infty)$ be a non-negative $\mu_1$-integrable Borel function on $\partial X$. Then the formula*

$$h(\alpha) = \int_B \mathsf{k}_\beta(\alpha)\varphi(\beta)\,\mu_1(d\beta) \tag{1.6}$$

*defines a harmonic function $h(\alpha)$ such that*

$$\lim_{n\to\infty} h(\beta_n) = \varphi(\beta_\infty) \quad \mathsf{P}_\theta\text{-a.s.} \qquad \text{and} \qquad \mathsf{E}_\alpha\,\varphi(\beta_\infty) = h(\alpha). \tag{1.7}$$

*Let $h(\alpha)$ be a bounded harmonic function. Then there is a bounded Borel function $\varphi(\beta_\infty)$ on $B$, such that* (1.6) *and* (1.7) *hold.*

### 1.2. The process

In this paper we consider the Markov chains $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, that are two-dimensional random walks. They are characterised by the properties P1, P2, P3 below.

**P1** Their state spaces are

$$\begin{aligned} \mathbf{Z}^2 &= \{(i,j),\ i,j \text{ are integers}\}, \\ \mathbf{Z}_+ \times \mathbf{Z} &= \{(i,j),\ i,j \text{ are integers},\ j \geq 0\}, \\ \mathbf{Z}_+^2 &= \{(i,j),\ i,j \text{ are integers},\ i,j \geq 0\} \end{aligned}$$

respectively.

**P2** The random walks are *maximally state homogeneous.* This property means that the state space can be represented as the union of a finite number of non-intersecting classes

$$X = \bigcup_r S_r,$$

such that for each $r$ and all $\alpha, \beta \in S_r$

$$p_{\beta,\beta+(i,j)} = p_{\alpha,\alpha+(i,j)}.$$

The latter probabilities will be denoted by ${}^{(r)}p_{ij}$.

The state space of the chain $\mathcal{L}_1$ is the union

$$\mathbf{Z}^2 = S \bigcup \Big( \bigcup_{m=1}^{n} S^m \Big),$$

where the sets $S^1, S^2, \ldots, S^n$ are *finite*. The transition probabilities ${}^{(r)}p_{ij}$ will be denoted by $p_{ij}$ and ${}^{(1)}p_{ij}, {}^{(2)}p_{ij}, \ldots, {}^{(n)}p_{ij}$ respectively.

The state space of the chain $\mathcal{L}_2$ is the union of two classes

$$\mathbf{Z}_+ \times \mathbf{Z} = S \cup S^{'},$$

where

$$\begin{cases} S &= \{(i,j) : j > 0\} \\ S^{'} &= \{(i,0)\}. \end{cases}$$

The part $S^{'}$ is called the $x$-axis. The probabilities ${}^{(r)}p_{ij}$ are denoted by $p_{ij}$ and $p'_{ij}$ according to their respective regions $S$ and $S^{'}$.

The state space of the chain $\mathcal{L}_3$ is divided into four classes:

$$\mathbf{Z}_+^2 = S \cup S^{'} \cup S^{''} \cup \{(0,0)\},$$

where

$$\begin{cases} S &= \{(i,j) : i,j > 0\}, \\ S^{'} &= \{(i,0) : i > 0\}, \\ S^{''} &= \{(0,j) : j > 0\}. \end{cases}$$

The internal parts $S^{'}$ and $S^{''}$ are called the $x$-axis and the $y$-axis. The probabilities ${}^{(r)}p_{ij}$ are denoted by $p_{ij}$, $p'_{ij}$, $p''_{ij}$, and $p^0_{ij}$ respectively.

**P3** (*Boundedness of the jumps*). For any $\alpha \in S_r$,

$$p_{\alpha,\beta} \neq 0 \quad \text{only for} \quad -1 \leq (\beta - \alpha)_i \leq 1,$$

where $(\beta - \alpha)_i$ is the $i$th coordinate of the vector $(\beta - \alpha)$, $i = 1, 2$.

In addition the next assumption will hold for all the chains $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ throughout the paper: the probabilities $p_{10}$, $p_{-10}$, $p_{01}$, $p_{0-1}$ for the class $S_r = S$ are non-zero and all other jump probabilities for this class equal zero.

We shall consider only *irreducible aperiodic* random walks. Assume additionally that the classes $S^{'}$, $S^{''}$ can be left in one jump with non-zero probability, i.e. $p'_{-11} + p'_{01} + p'_{11} > 0$, $p''_{1-1} + p''_{10} + p''_{11} > 0$.

We will not specify the choice of the initial state, since it influences neither transience, nor the Martin boundary.

Let us introduce the mean jump vectors

$$\begin{cases} \mathsf{E} &= (\mathsf{E}_x, \mathsf{E}_y) &= \Big(\sum_{i,j} i p_{ij}, \sum_{i,j} j p_{ij}\Big), \\ \mathsf{E}^{'} &= (\mathsf{E}^{'}_x, \mathsf{E}^{'}_y) &= \Big(\sum_{i,j} i p'_{ij}, \sum_{i,j} j p'_{ij}\Big), \\ \mathsf{E}^{''} &= (\mathsf{E}^{''}_x, \mathsf{E}^{''}_y) &= \Big(\sum_{i,j} i p''_{ij}, \sum_{i,j} j p''_{ij}\Big). \end{cases}$$

Throughout the paper we will assume that $\mathsf{E}_x \neq 0$, $\mathsf{E}_y \neq 0$ for $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. This implies transience of the chain $\mathcal{L}_1$. We restrict our attention only to transient Markov chains. Let us formulate the conditions for transience of the chains $\mathcal{L}_2$ and $\mathcal{L}_3$. All of these have been proved in [3].

**Theorem 1.7.** *Let* $\mathsf{E}_x \neq 0$, $\mathsf{E}_y \neq 0$. *The chain* $\mathcal{L}_2$ *is transient if and only if one of the following conditions holds:*

1) $\mathsf{E}_y > 0$;

2) $\mathsf{E}_y < 0, \quad \mathsf{E}_x \mathsf{E}_y^{'} - \mathsf{E}_y \mathsf{E}_x^{'} \neq 0.$

**Theorem 1.8.** *Let* $\mathsf{E}_x \neq 0$, $\mathsf{E}_y \neq 0$. *The chain* $\mathcal{L}_3$ *is transient if and only if one of the following conditions holds:*

1) $\mathsf{E}_x < 0, \quad \mathsf{E}_y < 0, \quad \mathsf{E}_x \mathsf{E}_y^{'} - \mathsf{E}_y \mathsf{E}_x^{'} > 0, \quad \mathsf{E}_y \mathsf{E}_x^{''} - \mathsf{E}_x \mathsf{E}_y^{''} \leq 0;$

2) $\mathsf{E}_x < 0, \quad \mathsf{E}_y < 0, \quad \mathsf{E}_x \mathsf{E}_y^{'} - \mathsf{E}_y \mathsf{E}_x^{'} \leq 0, \quad \mathsf{E}_y \mathsf{E}_x^{''} - \mathsf{E}_x \mathsf{E}_y^{''} > 0;$

3) $\mathsf{E}_x < 0, \quad \mathsf{E}_y < 0, \quad \mathsf{E}_x \mathsf{E}_y^{'} - \mathsf{E}_y \mathsf{E}_x^{'} > 0, \quad \mathsf{E}_y \mathsf{E}_x^{''} - \mathsf{E}_x \mathsf{E}_y^{''} > 0;$

4) $\mathsf{E}_x > 0, \quad \mathsf{E}_y < 0, \quad \mathsf{E}_x \mathsf{E}_y^{'} - \mathsf{E}_y \mathsf{E}_x^{'} > 0;$

5) $\mathsf{E}_x < 0, \quad \mathsf{E}_y > 0, \quad \mathsf{E}_y \mathsf{E}_x^{''} - \mathsf{E}_x \mathsf{E}_y^{''} > 0;$

6) $\mathsf{E}_x > 0, \quad \mathsf{E}_y > 0.$

## 2. Main results

### 2.1. Preliminaries

This subsection contains the necessary definitions and statements in order to present our results on the Martin boundary.

Let us choose the reference measure $\eta(\alpha)$ to be the $\delta$-measure at the origin $\alpha = (0;0)$.

We will restrict ourselves to the following cases:

(a) $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$;

(b) $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$.

Let $X(y)$ and $Y(x)$ be the algebraic functions determined by the equation

$$Q(x,y) \stackrel{\text{def}}{=} xy\big(1 - p_{10}x - p_{-10}x^{-1} - p_{01}y - p_{0-1}y^{-1}\big) = 0. \qquad (2.1)$$

**Lemma 2.1.** *The functions* $Y(x)$ *and* $X(y)$ *have four branch points* $x_1$, $x_2$, $x_3$, $x_4$ *and* $y_1$, $y_2$, $y_3$, $y_4$. *In case of* (a) *and* (b) *they satisfy the following inequalities:*

$$0 < x_1 < x_2 < 1 < x_3 < x_4, \qquad 0 < y_1 < y_2 < 1 < y_3 < y_4.$$

**Lemma 2.2.** *The Riemann surfaces of the functions* $Y(x)$ *and* $X(y)$ *are conformally equivalent and have genus* 1.

This means that the Riemann surfaces of $Y(x)$ and $X(y)$ are homeomorphic to a torus. We study the Riemann surface $\mathbf{S}$ for both $X(y)$ and $Y(x)$ with two different branched coverings:

$$h_x : \mathbf{S} \to \mathbf{P}_x, \qquad h_y : \mathbf{S} \to \mathbf{P}_y,$$

where $\mathbf{P}_x$ and $\mathbf{P}_y$ are the complex spheres of the variables $x$ and $y$ respectively. One can see $\mathbf{S}$ in Figure 2.1 in both cases (a) and (b). Any function $f$ on a domain $\mathcal{D} \subset \mathbf{P}_x$ can be lifted onto $h_x^{-1}(\mathcal{D}) \subset \mathbf{S}$. This yields a new function $f \circ h_x$, so that we are entitled to write

$$x(s) \stackrel{\text{def}}{=} h_x(s), \quad y(s) \stackrel{\text{def}}{=} h_y(s), \quad s \in \mathbf{S}.$$

Clearly, $Q(x(s), y(s)) \equiv 0$. We will sometimes write the pair $(x, y)$ to define a unique point $s \in \mathbf{S}$ such that $x(s) = x$, $y(s) = y$.

**Lemma 2.3.** *The set $S_r = \{s \in \mathbf{S} : x(s)$ and $y(s)$ are real or $\infty\}$ consists of two non-intersecting closed analytic curves $F_0$ and $F_1$ homological to one of the elements of the normal homology basis on $\mathbf{S}$. This element is different from $h_x^{-1}\{x : |x| = 1\}$. The curves $F_0$ and $F_1$ have the following properties:*

$$\begin{aligned} F_0 &= \{s : x_2 \le x(s) \le x_3\} = \{s : y_2 \le y(s) \le y_3\}, \\ F_1 &= \{s : x(s) \le 0 \text{ or } y(s) \le 0\} \cup \{s : x(s) = \infty \text{ or } y(s) = \infty\}. \end{aligned}$$

Let us mark the following points $s_1$, $s_2$, $s_3$, $s_4$ on $F_0$:

$$\begin{array}{ll} x(s_1) = x_3, & y(s_1) = \sqrt{p_{0-1}/p_{01}}, \\ x(s_2) = \sqrt{p_{-10}/p_{10}}, & y(s_2) = y_3, \\ x(s_3) = x_2, & y(s_3) = \sqrt{p_{0-1}/p_{01}}, \\ x(s_4) = \sqrt{p_{-10}/p_{10}}, & y(s_4) = y_2. \end{array}$$

Choose on $F_0$ the direction in order of the indices $s_i$ with the initial point $s_1$. We will consider throughout the paper the directed segments $[s', s''] \subset F_0$, $s' \le s \le s''$ (possible $s' = s''$) with respect to this choice, see Figure 2.1.

Next, we need to analyse the critical points of the function

$$\chi_\gamma(s) = |x(s) y^{\operatorname{tg}\gamma}(s)|, \qquad 0 \le \gamma < \pi$$

on $\chi^{-1}(0; \infty)$ in the sense of Morse theory, see [11]. For $\gamma = \pi/2$, put $\chi_\gamma(s) = |y(s)|$.

**Lemma 2.4.**

1. *For all fixed $\gamma \in [0; \pi)$, $\gamma \ne \pi/4, 3\pi/4$, the function $\chi_\gamma(s) = |x(s) y^{\operatorname{tg}\gamma}(s)|$ has four non-degenerate critical points $s_i(\gamma)$, $i = 1, 2, 3, 4$, on $\chi_\gamma^{-1}(0; \infty)$. They are such that*

$$\chi_\gamma(s_1(\gamma)) < \chi_\gamma(s_2(\gamma)) < \chi_\gamma(s_3(\gamma)) < \chi_\gamma(s_4(\gamma)).$$

*For $\gamma = \pi/4, 3\pi/4$ the function has two non-degenerate critical points $s_2$ and $s_3$, $\chi_\gamma(s_2) < \chi_\gamma(s_3)$.*

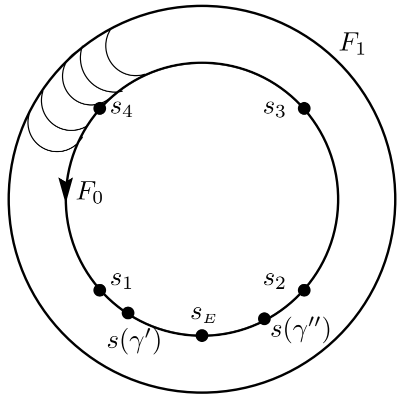


Figure 2.1(a). $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$.

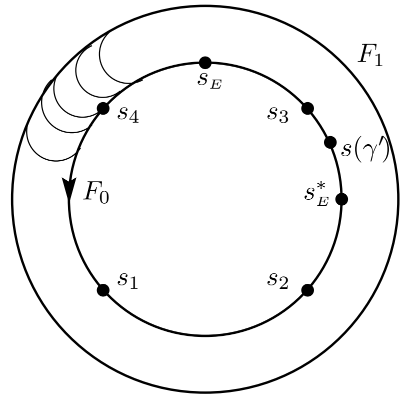


Figure 2.1(b). $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$.

2. *For all* $\gamma \in [0;\pi)$ *we have* $s_2(\gamma), s_3(\gamma) \in F_0$, $s_1(\gamma), s_4(\gamma) \in F_1$.

3. *For* $\gamma = 0, \pi/2$ *we have* $x(s_i(0)) = x_i$, $y(s_i(\pi/2)) = y_i$ *where* $x_i, y_i$ *are branch points of* $Y(x)$ *and* $X(y)$ *respectively,* $i = 1,2,3,4$. *For* $\gamma \neq 0, \pi/2$, *the values of* $x_i(s(\gamma)), y_i(s(\gamma))$ *can be found from the system of equations*

$$\begin{aligned} \operatorname{tg}\gamma &= \frac{p_{01}y - p_{0-1}/y}{p_{10}x - p_{-10}/x}, \qquad (2.2) \\ Q(x,y) &= 0. \end{aligned}$$

4. *For* $\gamma = 0, \pi/2$, *we have* $s_3(0) = s_1$, $s_2(0) = s_3$, $s_3(\pi/2) = s_2$, $s_2(\pi/2) = s_4$. *The functions* $s_2(\gamma)$ *and* $s_3(\gamma)$ *are continuous and strictly increasing on* $[0;\pi]$ *with ranges* $[s_3, s_1]$ *and* $[s_1, s_3]$ *respectively.*

Let us define the function $s(\gamma)$ on the segment $[0, 2\pi]$ as follows: $s(2\pi) := s(0)$,

$$s(\gamma) = \begin{cases} s_3(\gamma), & 0 \le \gamma < \pi; \\ s_2(\gamma - \pi), & \pi \le \gamma < 2\pi. \end{cases} \qquad (2.3)$$

**Corollary 2.1.** *The function* $s(\gamma)$ *is a homeomorphism between the segment* $[0;2\pi]$ *with the identified ends and the curve* $F_0$ *on the Riemann surface* $\mathbf{S}$.

Throughout the paper we denote for shortness $x(s(\gamma))$ by $x(\gamma)$ and $y(s(\gamma))$ by $y(\gamma)$.

*Remark* 2.1. Let $\gamma_E$ be the angle between the mean vector $(\mathsf{E}_x, \mathsf{E}_y)$ and the positive direction of the $x$-axis $\{j = 0\}$. Then $x(\gamma_E) = 1$, $y(\gamma_E) = 1$ and the associated point $s(\gamma_E)$ lies on $(s_1, s_2)$ if $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$ and on $(s_3, s_4)$ if $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$. We denote this point by $s_E$, see Figure 2.1.

We will also write $\pi_{ij}^{i_0 j_0}$ to denote the mean number of visits to state $(i,j)$ starting from $(i_0, j_0)$. Similarly $\mathsf{k}_{ij}(i_0, j_0) := \mathsf{k}_{(i,j)}(i_0, j_0)$.

### 2.2. Random walk in $\mathbf{Z}^2$

This subsection is devoted to the Martin boundary of the chain $\mathcal{L}_1$. Let

$$q_m(x,y) = \sum_{i,j} {}^{(m)}p_{ij}x^iy^j - 1, \qquad m = 1,\dots,n, \tag{2.4}$$

$$f_*^{i_0j_0}(x,y) = \sum_{m=1}^{n} q_m(x,y) \sum_{(i,j)\in S^m} \pi_{ij}^{i_0j_0} x^iy^j. \tag{2.5}$$

**Theorem 2.1.** *Let* $(i,j) \in \mathbf{Z}^2$. *Let* $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ *and let* $\gamma(r) \to \gamma$ *as* $r \to \infty$, *where* $0 \le \gamma \le 2\pi$. *Then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = \frac{x^{i_0}(\gamma)y^{j_0}(\gamma) + f_*^{i_0j_0}(x(\gamma),y(\gamma))}{1 + f_*^{00}(x(\gamma),y(\gamma))}. \tag{2.6}$$

*The Martin boundary of the chain* $\mathcal{L}_1$ *is homeomorphic to the curve* $F_0$ *on the Riemann surface* $\mathbf{S}$, *that is the circle* $[0,2\pi]$, *see Figure* 2.2. *The homeomorphism* $I$ *can be established by the mapping* $I : \gamma \to s(\gamma)$.

*The minimal Martin boundary is the same.*

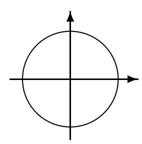

Figure 2.2

**Example** ($S^1, S^2, \dots, S^n$ are empty). If all sets $S^1, S^2, \dots, S^n$ are empty, then by the previous theorem

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = x^{i_0}(\gamma)y^{j_0}(\gamma). \tag{2.7}$$

This result was obtained by Ney and Spitzer [12]. We will briefly discuss their approach and its relation to ours. In [12] an irreducible homogeneous random walk on $\mathbf{Z}^d$ is considered, $d > 1$, with

$$p_{\alpha,\beta} = p_{0,\beta-\alpha} \qquad \text{for all } \alpha,\beta \in \mathbf{Z}^d, \tag{2.8}$$

$$\mathsf{E} = \sum_{\alpha\in Z^d} \alpha p_{0,\alpha} \neq 0.$$

Define the real-valued function $\Phi$ on $\mathbf{R}^d$ by

$$\Phi(u) = \sum_{\alpha\in Z^d} p_{0,\alpha} \exp(\alpha\cdot u).$$

Let

$$D = \{u \mid \Phi(u) \le 1\}, \qquad \partial D = \{u \mid \Phi(u) = 1\}.$$

For a random walk with bounded jumps, the mapping

$$u \to \frac{\operatorname{grad}\Phi(u)}{|\operatorname{grad}\Phi(u)|}$$

determines a homeomorphism between $\partial D$ and $\partial S = \{u: \ |u| = 1\}$.

**Theorem** ( [12]). *Let $\beta_n$ be a sequence of states in $\mathbf{Z}^d$ such that $\beta_n/|\beta_n| \to p$, for some $p \in \partial S$. Let $u$ be a unique solution of the equation*

$$p = \frac{\operatorname{grad}\Phi(u)}{|\operatorname{grad}\Phi(u)|}. \tag{2.9}$$

*Then for any $\alpha \in \mathbf{Z}^d$*

$$\lim_{n\to\infty} \mathsf{k}_{\beta_n}(\alpha) = \exp(u\cdot\alpha). \tag{2.10}$$

In our case $d = 2$ and

$$\Phi(u_1, u_2) = p_{10}\exp(u_1) + p_{01}\exp(u_2) + p_{-10}\exp(-u_1) + p_{0-1}\exp(-u_2).$$

If one puts

$$x = \exp(u_1), \qquad y = \exp(u_2), \tag{2.11}$$

then the set

$$\partial D = \{(u_1, u_2) \in \mathbf{R}^2 \mid \Phi(u) = 1\}$$

is homeomorphic to the set

$$\{(x, y) \in \mathbf{R}^2 \mid Q(x, y) = 0, x, y > 0\},$$

which in turn is homeomorphic to the “real circle” $F_0$ on our Riemann surface $\mathbf{S}$. Moreover, substituting (2.11) into (2.9) gives exactly equation (2.2) for our critical points, where $\gamma$ is the angle between the vector $p = (p_1, p_2)$ and the positive direction of the $x$-axis, i.e. $\operatorname{tg}\gamma = p_2/p_1$. (There are two roots of (2.2), for $x, y > 0$, i.e. two critical points on $F_0$. We have chosen one of these in (2.3). In equation (2.9) this has been provided for by the direction of $p$.) Thus relation (2.11) connects (2.9)–(2.10) to our result (2.7).

The method suggested in [12] is the following. In case of $p = \mathsf{E}/|\mathsf{E}|$, it is shown via the local central limit theorem that the asymptotics of the Green function is $\pi^{\alpha}_{\beta_n} \sim C\, n^{(1-d)/2}$. Hence, $\mathsf{k}_{\beta_n}(\alpha) \to 1$ for all $\alpha \in \mathbf{Z}^d$. Clearly, in this case the solution of equation (2.9) is given by $u = 0$. If $p \neq \mathsf{E}/|\mathsf{E}|$, one changes the probability measure in such a way that $p$ is the corresponding normed drift vector. To this end, one should determine the solution $u$ of (2.9) for a given $p$ and then put

$${}^{u}p_{\alpha,\beta} = p_{\alpha,\beta}\exp(u\cdot(\beta-\alpha)) \qquad \text{for all } \alpha, \beta \in \mathbf{Z}^d. \tag{2.12}$$

As a consequence, we have

$${}^{u}\mathsf{E} = \sum_{\alpha\in Z^d} \alpha\, {}^{u}p_{0,\alpha} = \operatorname{grad}\, \Phi(u) \tag{2.13}$$

and $p = {}^u\mathsf{E}/|{}^u\mathsf{E}|$. By the above case ${}^u\pi^\alpha_{\beta_n} \sim C(u)\, n^{(1-d)/2}$. Then the following important expression

$$\begin{aligned} {}^u\pi^\alpha_\beta = \sum_{n=0}^\infty {}^u p^n_{\alpha,\beta} &= \sum_{n=0}^\infty p^n_{\alpha,\beta} \exp(u\cdot(\beta-\alpha)) \\ &= \pi^\alpha_\beta \exp(u\cdot(\beta-\alpha)) \end{aligned} \tag{2.14}$$

implies that $\pi^\alpha_{\beta_n} \sim C(u)\, n^{(1-d)/2} \exp(u\cdot(\alpha-\beta_n))$. Thus $\mathsf{k}_{\beta_n}(\alpha) \to \exp(u\cdot\alpha)$.

This method relies on relation (2.14), which holds only if $p_{\alpha,\beta} = p_{0,\beta-\alpha}$ for *all* $\alpha, \beta \in \mathbf{Z}^d$, i.e. $S^1, \ldots, S^n$ are empty. It fails whenever the transition probabilities are "spoiled" even at one point of the space. We propose another method, which remains valid, even if the jump probabilities in some points of the state space are changed.

### 2.3. Random walk in $\mathbf{Z}_+ \times \mathbf{Z}$, $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$

In this subsection we will formulate our results on the Martin boundary for the chain $\mathcal{L}_2$ under the assumption $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$. Let

$$q(x,y) = x\Big(\sum_{i,j} p'_{ij} x^i y^j - 1\Big). \tag{2.15}$$

**Lemma 2.5.** *The system of equations*

$$\begin{cases} Q(x,y) = 0 \\ q(x,y) = 0 \end{cases} \tag{2.16}$$

*has a solution* $(x', y')$ *satisfying*

$$\begin{cases} 1 < x' < x_3 \\ p_{0-1}/p_{01} < y' < \sqrt{p_{0-1}/p_{01}} \end{cases} \tag{2.17}$$

*if and only if* $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$.

*The system* (2.16) *has a solution* $(x'', y'')$ *satisfying*

$$\begin{cases} x_2 < x'' < 1 \\ p_{0-1}/p_{01} < y'' < \sqrt{p_{0-1}/p_{01}} \end{cases} \tag{2.18}$$

*if and only if* $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$.

*For* $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$ [ $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$ ] *the solution of* (2.16) *satisfying* (2.17) [*resp.* (2.18)] *is unique.*

For $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$ and $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$ let us define the angles $\gamma'$, $0 < \gamma' < \pi$, and $\gamma''$, $0 < \gamma'' < \pi$, such that

$$x(\gamma') = x', \qquad y(\gamma') = \frac{p_{0-1}}{p_{01} y'}, \tag{2.19}$$

$$x(\gamma'') = x'', \qquad y(\gamma'') = \frac{p_{0-1}}{p_{01} y''}. \tag{2.20}$$

By virtue of Lemma 2.4

$$\mathrm{tg}\,\gamma' = \frac{p_{0-1}/y' - p_{01}y'}{p_{10}x' - p_{-10}/x'}, \qquad \mathrm{tg}\,\gamma'' = \frac{p_{0-1}/y'' - p_{01}/y''}{p_{10}x'' - p_{0-1}/x''}.$$

Moreover, we have $0 < \gamma' < \gamma_E < \gamma'' < \pi$ and

$$\begin{aligned} s(\gamma') &= (x'\,,\, p_{0-1}/(p_{01}y')) \in (s_1; s_E), \\ s(\gamma'') &= (x''\,,\, p_{0-1}/(p_{01}y'')) \in (s_E; s_3), \end{aligned}$$

see Figure 2.1(a).

**Theorem 2.2.** *Let $(i,j) \in \mathbf{Z}_+ \times \mathbf{Z}$. Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and $\gamma(r) \to \gamma$ as $r \to \infty$, where $0 \le \gamma \le \pi$.*

1. *If $q(x_3, \sqrt{p_{0-1}/p_{01}}) < 0$, $q(x_2, \sqrt{p_{0-1}/p_{01}}) < 0$, then for all $\gamma \in [0,\pi]$*

$$\begin{aligned} \lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) \;=\; & \big[\, x^{i_0}(\gamma)\, y^{j_0}(\gamma)\, q\big(x(\gamma),\ p_{0-1}/(p_{01}y(\gamma))\big) \\ & \quad - x^{i_0}(\gamma)\,\big(p_{0-1}/(p_{01}y(\gamma))\big)^{j_0} q\big(x(\gamma),\ y(\gamma)\big)\,\big] \\ & \times \big[\, q\big(x(\gamma),\ p_{0-1}/(p_{01}y(\gamma))\big) - q\big(x(\gamma),\ y(\gamma)\big)\,\big]^{-1}. \end{aligned} \tag{2.21}$$

*The Martin boundary is homeomorphic to the segment $[s_1, s_3]$ on $F_0$, that is to the arc $[0,\pi]$, see Figure* 2.3(a). *This homeomorphism is given by the mapping $I : \gamma \to s(\gamma)$.*

*The minimal Martin boundary is the same.*

2. *If $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$, $q(x_2, \sqrt{p_{0-1}/p_{01}}) < 0$, then one can define the pair $(x', y')$ by Lemma* 2.5 *and the angle $\gamma'$ by* (2.19). *For $\gamma \in [0, \gamma']$*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = (x')^{i_0}(y')^{j_0}. \tag{2.22}$$

*For $\gamma \in (\gamma', \pi]$ the asymptotics of the Martin kernel is given by* (2.21).

*The Martin boundary is homeomorphic to the segment $[s(\gamma'), s_3]$ on $F_0$, that is to the arc $[\gamma', \pi]$, see Figure 2.3(b). This homeomorphism is given by the mapping $I : \gamma \to s(\gamma)$.*

*The minimal Martin boundary is the same.*

3. *If $q(x_3, \sqrt{p_{0-1}/p_{01}}) < 0$, $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$, then one can define the pair $(x'', y'')$ by Lemma* 2.5 *and the angle $\gamma''$ by* (2.20). *For $\gamma \in [\gamma'', \pi]$*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = (x'')^{i_0}(y'')^{j_0}. \tag{2.23}$$

*For $\gamma \in [0, \gamma'')$ the asymptotics of the Martin kernel is given by* (2.21).

*The Martin boundary is homeomorphic to the segment $[s_1, s(\gamma'')]$ on $F_0$, that is to the arc $[0, \gamma'']$, see Figure* 2.3(c). *This homeomorphism is given by the mapping $I : \gamma \to s(\gamma)$.*

*The minimal Martin boundary is the same.*

4. *If $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$, $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$, then one can define the pairs $(x', y')$ and $(x'', y'')$ by Lemma* 2.5 *and the angles $0 < \gamma' < \gamma'' < \pi$*

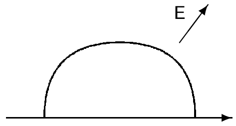


Figure 2.3(a)

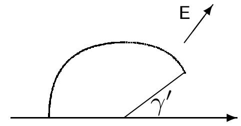


Figure 2.3(b)

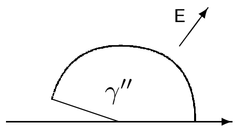


Figure 2.3(c)

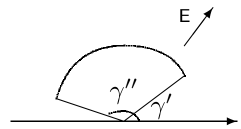


Figure 2.3(d)

*by* (2.19), (2.20). *The asymptotics of the Martin kernel is given by* (2.22) *for* $\gamma \in [0, \gamma']$, *by* (2.21) *for* $\gamma \in (\gamma', \gamma'')$, *and by* (2.23) *for* $\gamma \in [\gamma'', \pi]$.

*The Martin boundary is homeomorphic to the segment* $[s(\gamma'), s(\gamma'')]$ *on* $F_0$, *that is to the arc* $[\gamma', \gamma'']$, *see Figure* 2.3(d). *This homeomorphism is given by the mapping* $I : \gamma \to s(\gamma)$.

*The minimal Martin boundary is the same.*

### 2.4. Random walk in $\mathbf{Z}_+ \times \mathbf{Z}$, $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$

In this subsection we describe the Martin boundary for the chain $\mathcal{L}^2$ under the following assumptions:

- $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$;
- $\mathsf{E}_x \mathsf{E}_y^{'} - \mathsf{E}_y \mathsf{E}_x^{'} > 0$.

Let us define the angle $\gamma_E^* \in (0, \pi)$ by

$$x(\gamma_E^*) = 1, \qquad y(\gamma_E^*) = \frac{p_{0-1}}{p_{01}}. \tag{2.24}$$

Note that $\gamma_E^* = \gamma_E - \pi/2$. (In fact, by virtue of Lemma 2.4, $\operatorname{tg} \gamma_E^* = (p_{-10} - p_{10})/(p_{01} - p_{0-1})$.) Then $\pi/2 < \gamma_E^* < \pi$ and $s_E^* := s(\gamma_E^*) = (1, p_{0-1}/p_{01}) \in (s_2, s_3)$, see Figure 2.1(b).

The function $q(x, y)$ is defined by (2.15).

**Lemma 2.6.** *The system of equations*

$$\left\{ \begin{array}{lcl} Q(x,y) & = & 0 \\ q(x,y) & = & 0 \end{array} \right. \tag{2.25}$$

*has a solution* $(x', y')$ *satisfying*

$$\left\{ \begin{array}{l} x_2 < x' < 1 \\ 1 < y' < \sqrt{p_{0-1}/p_{01}} \end{array} \right. \tag{2.26}$$

*if and only if* $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$.

*The solution* $(x', y')$ *of* (2.25) *satisfying* (2.26) *is unique.*

If $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$, one can define the angle $\gamma' \in (0, \pi)$ by

$$x(\gamma') = x', \qquad y(\gamma') = \frac{p_{0-1}}{p_{01}y'}. \tag{2.27}$$

By virtue of Lemma 2.4

$$\operatorname{tg}\gamma' = \frac{p_{0-1}/y' - p_{01}y'}{p_{10}x' - p_{0-1}/x'}.$$

Moreover, $\pi/2 < \gamma_{_E}^* < \gamma' < \pi$ and

$$s(\gamma') = (x', p_{0-1}/(p_{01}y')) \in (s_{_E}^*, s_3) \in (s_2, s_3),$$

see Figure 2.1(b).

**Theorem 2.3.** *Let $(i,j) \in \mathbf{Z} \times \mathbf{Z}_+$. Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and $\gamma(r) \to \gamma$ as $r \to \infty$, where $0 \le \gamma \le \pi$.*

1. *If $q(x_2, \sqrt{p_{0-1}/p_{01}}) < 0$, then for $\gamma \in [0, \gamma_{_E}^*]$*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = 1 \tag{2.28}$$

*and for $\gamma \in (\gamma_{_E}^*, \pi]$*

$$\begin{aligned} \lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) \quad = \quad & \big[\, x^{i_0}(\gamma)\, y^{j_0}(\gamma)\, q\big(x(\gamma),\ p_{0-1}/(p_{01}y(\gamma))\big) \\ & - x^{i_0}(\gamma)\, \big(p_{0-1}/(p_{01}y(\gamma))\big)^{j_0}\, q\big(x(\gamma),\ y(\gamma)\big)\,\big] \\ & \times \big[\, q\big(x(\gamma),\ p_{0-1}/(p_{01}y(\gamma))\big) - q\big(x(\gamma),\ y(\gamma)\big)\,\big]^{-1}. \end{aligned} \tag{2.29}$$

*The Martin boundary is homeomorphic to the segment $[s_{_E}^*, s_3]$ on $F_0$, that is to the arc $[\gamma_{_E}^*, \pi]$, see Figure 2.4(a). This homeomorphism is given by the mapping $I : \gamma \to s/(\gamma)$.*

*The minimal Martin boundary is the same.*

2. *If $q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$, then one can define the pair $(x', y')$ by Lemma 2.6 and the angle $\gamma'$ as in (2.27). For $\gamma \in [0, \gamma_{_E}^*]$ the asymptotics of the Martin kernel is given by (2.28) and for $\gamma \in (\gamma_{_E}^*, \gamma')$ it is given by (2.29). For $\gamma \in [\gamma', \pi]$:*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = (x')^{i_0}(y')^{j_0}. \tag{2.30}$$

*The Martin boundary is homeomorphic to the segment $[s_{_E}^*, s(\gamma')]$ on $F_0$, that is to the arc $[\gamma_{_E}^*, \gamma']$, see Figure 2.4(b). This homeomorphism is given by the mapping $I : \gamma \to s(\gamma)$.*

*The minimal Martin boundary is the same.*

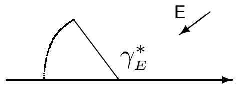


Figure 2.4(a)

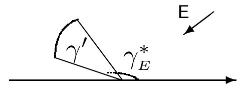


Figure 2.4(b)

### 2.5. Random walk in $\mathbf{Z}_+^2$, $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$

In this subsection we formulate the results on the Martin boundary for the chain $\mathcal{L}_3$ under the assumption $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$. Let

$$
\begin{aligned}
q(x, y) &= x\Big(\sum_{i,j} p'_{ij} x^i y^j - 1\Big), \\
\widetilde{q}(x, y) &= y\Big(\sum_{i,j} p''_{ij} x^i y^j - 1\Big), \\
q_0(x, y) &= \sum_{i,j} p^0_{ij} x^i y^j - 1.
\end{aligned} \tag{2.31}
$$

**Lemma 2.7.** *The system of equations*

$$
\left\{ \begin{array}{lcl} Q(x, y) &=& 0 \\ q(x, y) &=& 0 \end{array} \right. \tag{2.32}
$$

*has a solution $(x', y')$ satisfying*

$$
\left\{ \begin{array}{l} 1 < x' < x_3 \\ p_{0-1}/p_{01} < y' < \sqrt{p_{0-1}/p_{01}} \end{array} \right. \tag{2.33}
$$

*if and only if $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$.*

*The system of equations*

$$
\left\{ \begin{array}{lcl} Q(x, y) &=& 0 \\ \widetilde{q}(x, y) &=& 0 \end{array} \right. \tag{2.34}
$$

*has a solution $(x'', y'')$ satisfying*

$$
\left\{ \begin{array}{l} p_{-10}/p_{10} < x'' < \sqrt{p_{-10}/p_{10}} \\ 1 < y'' < y_3 \end{array} \right. \tag{2.35}
$$

*if and only if $\widetilde{q}(\sqrt{p_{-10}/p_{10}}, y_3) > 0$.*

*For $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$ [resp. $\widetilde{q}(\sqrt{p_{-10}/p_{10}}, y_3) > 0$] the solution of* (2.32) *[resp.* (2.34)*] satisfying* (2.33) *[resp.* (2.35)*] is unique.*

For $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$ and $\widetilde{q}(\sqrt{p_{-10}/p_{10}}, y_3) > 0$ let us define the angles $\gamma'$, $0 < \gamma' < \pi/2$, and $\gamma'' > 0$, $0 < \gamma'' < \pi/2$, such that

$$
x(\gamma') = x', \qquad y(\gamma') = \frac{p_{0-1}}{p_{01} y'}, \tag{2.36}
$$

$$
x(\gamma'') = \frac{p_{10}}{p_{-10} x''}, \qquad y(\gamma'') = y''. \tag{2.37}
$$

By virtue of Lemma 2.4

$$\operatorname{tg}\gamma' = \frac{p_{0-1}/y' - p_{01}y'}{p_{10}x' - p_{-10}/x'}, \qquad \operatorname{tg}\gamma'' = \frac{p_{01}y'' - p_{0-1}/y''}{p_{-10}/x'' - p_{10}x''}.$$

Moreover, we have $0 < \gamma' < \gamma_E < \gamma'' < \pi/2$ and

$$\begin{aligned} s(\gamma') &= (x' \, , \, p_{0-1}/(p_{01}y')) \in (s_1, s_E), \\ s(\gamma'') &= (p_{-10}/(p_{10}x'') \, , \, y'') \in (s_E, s_2), \end{aligned}$$

see Figure 2.1(a).

Let us introduce the generating functions

$$\pi^{i_0j_0}(x) = \sum_{i=1}^{\infty} \pi_{i0}^{i_0j_0} x^{i-1}, \qquad \widetilde{\pi}^{i_0j_0}(y) = \sum_{j=1}^{\infty} \pi_{0j}^{i_0j_0} y^{j-1} \tag{2.38}$$

in the discs $\{x : |x| < 1\}$ and $\{y : |y| < 1\}$ respectively.

**Proposition 2.1.** *We have*

$$\sum_{i=1}^{\infty} \pi_{i0}^{i_0j_0} < \infty, \qquad \sum_{j=1}^{\infty} \pi_{0j}^{i_0j_0} < \infty.$$

**Theorem 2.4.** *Let* $(i,j) \in \mathbf{Z}_+^2$. *Let* $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ *and let* $\gamma(r) \to \gamma$ *as* $r \to \infty$, *where* $0 \le \gamma \le \pi/2$.

1. *Assume that* $q(x_3, \sqrt{p_{0-1}/p_{01}}) < 0$, $\widetilde{q}(\sqrt{p_{-10}/p_{10}}, y_3) < 0$. *If* $\gamma \in [0, \gamma_E]$, *then* $\sqrt{p_{0-1}/p_{01}} < y(\gamma) < 1$ *and*

$$\begin{aligned}
&\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) \\
&= \Big[ q(x(\gamma) \, , \, p_{0-1}/(p_{01}y(\gamma))) \\
&\quad \times \left[ x^{i_0}(\gamma) y^{j_0}(\gamma) + q_0(x(\gamma), y(\gamma)) \pi_{00}^{i_0j_0} + \widetilde{q}(x(\gamma), y(\gamma)) \widetilde{\pi}^{i_0j_0}(y(\gamma)) \right] \\
&\quad - q(x(\gamma), y(\gamma)) \\
&\quad \times \Big[ x^{i_0}(\gamma) \big(p_{0-1}/(p_{01}y(\gamma))\big)^{j_0} + q_0(x(\gamma) \, , \, p_{0-1}/(p_{01}y(\gamma))) \pi_{00}^{i_0j_0} \\
&\qquad\qquad + \widetilde{q}(x(\gamma) \, , \, p_{0-1}/(p_{01}y(\gamma))) \widetilde{\pi}^{i_0j_0}(p_{0-1}/(p_{01}y(\gamma))) \Big] \Big] \\
&\times \Big[ q(x(\gamma) \, , \, p_{0-1}/(p_{01}y(\gamma))) \\
&\quad \times \left[ 1 + q_0(x(\gamma), y(\gamma)) \pi_{00}^{00} + q(x(\gamma), y(\gamma)) \widetilde{\pi}^{00}(y(\gamma)) \right] \\
&\quad - q(x(\gamma), y(\gamma)) \\
&\quad \times \Big[ 1 + q_0(x(\gamma) \, , \, p_{0-1}/(p_{01}y(\gamma))) \pi_{00}^{00} \\
&\qquad\qquad + q(x(\gamma) \, , \, p_{0-1}/(p_{01}y(\gamma))) \widetilde{\pi}^{00}(p_{0-1}/(p_{01}y(\gamma))) \Big] \Big]^{-1}.
\end{aligned} \tag{2.39}$$

*If* $\gamma \in [\gamma_E, \pi/2]$, *then* $p_{-10}/p_{01} < x(\gamma) < 1$ *and*

$$
\begin{aligned}
&\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) \qquad (2.40)\\
&\quad = \Big[\widetilde{q}(p_{-10}/(p_{10}x(\gamma))\,,\, y(\gamma))\\
&\qquad \times \big[x^{i_0}(\gamma)y^{j_0}(\gamma) + q_0(x(\gamma), y(\gamma))\pi_{00}^{i_0j_0} q(x(\gamma), y(\gamma))\pi^{i_0j_0}(x(\gamma))\big]\\
&\qquad - \widetilde{q}(x(\gamma), y(\gamma))\\
&\qquad \times \big[\big(p_{-10}/(p_{10}x(\gamma))\big)^{i_0} y^{j_0}(\gamma) + q_0(p_{-10}/(p_{10}y(\gamma))\,,\, y(\gamma))\pi_{00}^{i_0j_0}\\
&\qquad\qquad + q(p_{-10}/(p_{10}x(\gamma))\,,\, y(\gamma))\pi^{i_0j_0}(p_{-10}/(p_{10}x(\gamma)))\big]\Big]\\
&\quad \times \Big[\widetilde{q}(p_{-10}/(p_{10}x(\gamma))\,,\, y(\gamma))\\
&\qquad \times \big[1 + q_0(x(\gamma), y(\gamma))\pi_{00}^{00} q(x(\gamma), y(\gamma))\pi^{00}(x(\gamma))\big]\\
&\qquad - \widetilde{q}(x(\gamma), y(\gamma))\\
&\qquad \times \big[1 + +q_0(p_{-10}/(p_{10}y(\gamma))\,,\, y(\gamma))\pi_{00}^{00}\\
&\qquad\qquad + q(p_{-10}/(p_{10}x(\gamma))\,,\, y(\gamma))\pi^{00}(p_{-10}/(p_{10}x(\gamma)))\big]\Big]^{-1}.
\end{aligned}
$$

(*If* $\gamma = \gamma_E$, *then* $\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = 1$ *in agreement with* (2.39) *and* (2.40).)

*The Martin boundary is homeomorphic to the segment* $[s_1, s_2]$ *on* $F_0$, *that is to the arc* $[0, \pi/2]$, *see Figure* 2.5(a). *This homeomorphism is given by the mapping* $I : \gamma \to s(\gamma)$.

*The minimal Martin boundary is the same.*

2. *Assume that* $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$, $\widetilde{q}(\sqrt{p_{-10}/p_{10}}, y_3) < 0$. *One can define the pair* $(x', y')$ *by Lemma* 2.7 *and the angle* $\gamma'$ *by* (2.36)*;* $0 < \gamma' < \gamma_E$. *For* $\gamma \in [0, \gamma']$

$$
\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = \frac{(x')^{i_0}(y')^{j_0} + q_0(x', y')\pi_{00}^{i_0j_0} + \widetilde{q}(x', y')\widetilde{\pi}^{i_0j_0}(y')}{1 + q_0(x', y')\pi_{00}^{00} + \widetilde{q}(x', y')\widetilde{\pi}^{00}(y')}. \qquad (2.41)
$$

*For* $\gamma \in (\gamma', \pi/2]$ *the asymptotics of the Martin kernel is given by* (2.39) *whenever* $\gamma$ *is not greater than* $\gamma_E$ *and by* (2.40) *otherwise.*

*The Martin boundary is homeomorphic to the segment* $[s(\gamma'), s_2]$ *on* $F_0$, *that is to the arc* $[\gamma', \pi/2]$, *see Figure* 2.5(b). *This homeomorphism is given by the mapping* $I : \gamma \to s(\gamma)$.

*The minimal Martin boundary is the same.*

3. *Assume that* $q(x_3, \sqrt{p_{0-1}/p_{01}}) < 0$, $\widetilde{q}(\sqrt{p_{-10}/p_{10}}, y_3) > 0$. *One can define the pair* $(x'', y'')$ *by Lemma* 2.7 *and the angle* $\gamma''$ *by* (2.37)*;* $\gamma_E < \gamma'' < \pi/2$. *For* $\gamma \in [\gamma'', \pi/2]$

$$
\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = \frac{(x'')^{i_0}(y'')^{j_0} + q_0(x'', y'')\pi_{00}^{i_0j_0} + q(x'', y'')\pi^{i_0j_0}(x'')}{1 + q_0(x'', y'')\pi_{00}^{00} + q(x'', y'')\pi^{00}(x'')}. \qquad (2.42)
$$

*For* $\gamma \in [0, \gamma'')$ *the asymptotics of the Martin kernel is given by* (2.39) *if* $\gamma$ *is not greater than* $\gamma_E$ *and by* (2.40) *otherwise.*

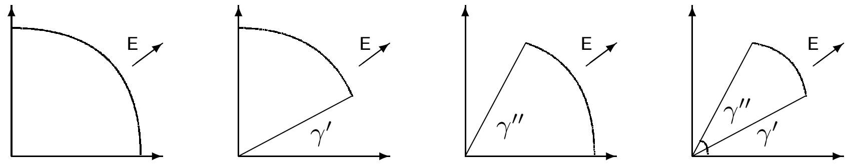


Figure 2.5(a) Figure 2.5(b) Figure 2.5(c) Figure 2.5(d)

*The Martin boundary is homeomorphic to the segment* $[s_1, s(\gamma'')]$ *on* $F_0$, *that is to the arc* $[0, \gamma'']$, *see Figure* 2.5(c). *This homeomorphism is given by the mapping* $I : \gamma \to s(\gamma)$.

*The minimal Martin boundary is the same.*

4. *Assume that* $q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$, $q(\sqrt{p_{-10}/p_{10}}, y_3) > 0$. *Then one can define the pairs* $(x', y')$ *and* $(x'', y'')$ *by Lemma* 2.7 *and the angles* $\gamma'$, $\gamma''$ *by* (2.36), (2.37). *The asymptotics of the Martin kernel is given by* (2.41) *for* $\gamma \in [0, \gamma']$, *by* (2.39) *for* $\gamma \in (\gamma', \gamma_E]$, *by* (2.40) *for* $\gamma \in [\gamma_E, \gamma'')$, *and by* (2.42) *for* $\gamma \in [\gamma'', \pi/2]$.

*The Martin boundary is homeomorphic to the segment* $[s(\gamma'), s(\gamma'')]$ *on* $F_0$, *that is to the arc* $[\gamma', \gamma'']$, *see Figure* 2.5(d). *This homeomorphism is given by the mapping* $I : \gamma \to s(\gamma)$.

*The minimal Martin boundary is the same.*

## 2.6. Random walk in $\mathbf{Z}_+^2$: $\mathsf{E}_x < 0, \mathsf{E}_y < 0$, escape to infinity along one axis

This subsection is devoted to the Martin boundary of the chain $\mathcal{L}_3$ under the following assumptions:

- $\mathsf{E}_x < 0, \mathsf{E}_y < 0$;
- $\mathsf{E}_x \mathsf{E}'_y - \mathsf{E}_y \mathsf{E}'_x > 0$;
- $\mathsf{E}_y \mathsf{E}''_x - \mathsf{E}_x \mathsf{E}''_y < 0$.

The functions $q(x, y)$, $\widetilde{q}(x, y)$ and $q_0(x, y)$ are the same as in the previous subsection.

**Lemma 2.8.** *The system of equations*

$$\left\{ \begin{array}{lll} Q(x,y) & = & 0 \\ \widetilde{q}(x,y) & = & 0 \end{array} \right. \tag{2.43}$$

*has a solution* $(x', y')$ *satisfying*

$$\left\{ \begin{array}{l} x_2 \le x' < 1 \\ 1 < y' < p_{0-1}/p_{01} \end{array} \right. \tag{2.44}$$

*if and only if* $\widetilde{q}(1, p_{0-1}/p_{01}) > 0$.

*This solution is unique.*

For $\widetilde{q}(1, p_{0-1}/p_{01}) > 0$ let us introduce the angle $\gamma_0 \in (0, \pi/2)$ by

$$\left(\frac{p_{-10}}{p_{10}x'}\right)^{\operatorname{ctg}\gamma_0} y' = \frac{p_{0-1}}{p_{01}}. \tag{2.45}$$

As in the previous subsection we have the generating functions

$$\pi^{i_0 j_0}(x) = \sum_{i=1}^{\infty} \pi_{i0}^{i_0 j_0} x^{i-1}, \qquad \widetilde{\pi}^{i_0 j_0}(y) = \sum_{j=1}^{\infty} \pi_{0j}^{i_0 j_0} y^{j-1}.$$

They are defined in the domains $\{x : |x| < 1\}$ and $\{y : |y| < 1\}$ respectively.

**Theorem 2.5.** *Let $(i, j) \in \mathbf{Z}_+^2$. Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and let $\gamma(r) \to \gamma$ as $r \to \infty$, where $0 \le \gamma \le \pi/2$.*

*1. Assume that $\widetilde{q}(1, p_{0-1}/p_{01}) < 0$. Then for all $\gamma \in [0, \pi/2]$*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = 1. \tag{2.46}$$

*The Martin boundary is trivial.*

*2. Assume that $\widetilde{q}(1, p_{-10}/p_{10}) > 0$. One can define the pair $(x', y')$ by Lemma 2.8 and the angle $\gamma_0$ by (2.45). If $\gamma \in [0, \gamma_0)$, then the asymptotics of the Martin kernel is given by (2.46). If $\gamma \in (\gamma_0, \pi/2]$, then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = \frac{C(i_0, j_0)}{C(0,0)}, \tag{2.47}$$

*where*

$$C(i, j) \quad = \quad (x')^i (y')^j + q_0(x', y')\pi_{00}^{ij} + q(x', y')\pi^{ij}(x').$$

*If $\gamma = \gamma_0$, then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0 j_0) = \lim_{r\to\infty} \frac{C_1\, C(i_0, j_0) + C_2\left(p_{-10}/(p_{10}x')\right)^{i-j\operatorname{ctg}\gamma_0}}{C_1\, C(0,0) + C_2\left(p_{-10}/(p_{10}x')\right)^{i-j\operatorname{ctg}\gamma_0}}, \tag{2.48}$$

*where*

$$C_1 \quad = \quad \frac{\widetilde{q}(p_{-10}/(p_{10}x')\,,\, y')\, \mathrm{res}_{y=y'}\, \widetilde{q}^{-1}(X(y), y)}{2p_{-10}y'/x' + p_{01}y'^2 + p_{0-1} - y'},$$

$$C_2 \quad = \quad \frac{q(1\,,\, p_{0-1}/p_{01})\, \mathrm{res}_{x=1}\, q^{-1}(x, Y(x))}{p_{0-1} - p_{01}}.$$

*(The branch $X(y)$ [resp. $Y(x)$] is such that $X(y') = x'$ [resp. $Y(1) = 1$].)*

*If $\operatorname{ctg}\gamma_0$ is irrational, then the Martin boundary is homeomorphic to the set $[-\infty, \infty]$. If $\operatorname{ctg}\gamma_0$ is rational, then the Martin boundary is homeomorphic to the set $\mathbf{Z} \cup \{\infty\} \cup \{-\infty\}$. This homeomorphism is given by $\lim_{r\to\infty}(i - j\operatorname{ctg}\gamma_0)$.*

*The minimal Martin boundary is homeomorphic to a two-points set. These points are determined by (2.46) and (2.47).*

### 2.7. Random walk in $Z_+^2$, $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$, escape to infinity along two axes

In this subsection we find the Martin boundary of the chain $\mathcal{L}_3$ under the following assumptions:

- $\mathsf{E}_x < 0, \mathsf{E}_y < 0$;
- $\mathsf{E}_x\,\mathsf{E}'_y - \mathsf{E}_y\,\mathsf{E}'_x > 0$;
- $\mathsf{E}_y\,\mathsf{E}''_x - \mathsf{E}_x\,\mathsf{E}''_y > 0$.

The functions $q(x,y)$, $\widetilde{q}(x,y)$, $q_\circ(x,y)$, $\pi^{i_0 j_0}(x)$, $\widetilde{\pi}^{i_0 j_0}(y)$ are the same as in Subsection 2.5.

**Lemma 2.9.** *There exist constants $C(i_0,j_0)$ and $\widetilde{C}(i_0,j_0)$ such that*

$$\pi_{i0}^{i_0 j_0} \quad\to\quad C(i_0,j_0) \qquad \text{as } i\to\infty, \tag{2.49}$$

$$\pi_{0j}^{i_0 j_0} \quad\to\quad \widetilde{C}(i_0,j_0) \qquad \text{as } j\to\infty. \tag{2.50}$$

Let us define the angle $\gamma_0 \in (0,\pi/2)$ by

$$\left(\frac{p_{10}}{p_{-10}}\right)^{\operatorname{ctg}\gamma_0} = \frac{p_{01}}{p_{0-1}}. \tag{2.51}$$

**Theorem 2.6.** *Let $(i,j)\in \mathbf{Z}_+^2$ be given by $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ where $\gamma(r)\to\gamma$ and $r\to\infty$, where $0\le\gamma\le\pi/2$. The angle $\gamma_0$ is defined by (2.51). If $\gamma\in[0,\gamma_0)$, then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = \frac{C(i_0,j_0)}{C(0,0)}. \tag{2.52}$$

*If $\gamma\in(\gamma_0,\pi/2]$, then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = \frac{\widetilde{C}(i_0,j_0)}{\widetilde{C}(0,0)}, \tag{2.53}$$

*where the constants $C(i_0,j_0)$ and $\widetilde{C}(i_0,j_0)$ are defined by Lemma 2.9.*

*If $\gamma=\gamma_0$, then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = \lim_{r\to\infty} \frac{C_1\,C(i_0,j_0) + C_2\,\widetilde{C}(i_0,j_0)(p_{10}/p_{-10})^{i-j\operatorname{ctg}\gamma_0}}{C_1\,C(0,0) + C_2\,\widetilde{C}(0,0)(p_{10}/p_{-10})^{i-j\operatorname{ctg}\gamma_0}}, \tag{2.54}$$

*where*

$$C_1 = q(p_{-10}/p_{10},1)/(p_{-10}-p_{10}), \qquad C_2 = \widetilde{q}(1,p_{0-1}/p_{01})/(p_{0-1}-p_{01}).$$

*If $\operatorname{ctg}\gamma_0$ is irrational, then the Martin boundary is homeomorphic to the set $[-\infty,+\infty]$. If $\operatorname{ctg}\gamma_0$ is rational, then the Martin boundary is homeomorphic to the set $\mathbf{Z}\cup\{\infty\}\cup\{-\infty\}$. The homeomorphism is given by $\lim_{r\to\infty}(i-j\operatorname{ctg}\gamma_0)$.*

*The minimal Martin boundary is homeomorphic to a two-points set. These points are determined by (2.52) and (2.53).*

## 3. Proofs

### 3.1. Preliminaries

In this subsection we give the general structure of the proofs of Theorems 2.1–2.6. We also prove all necessary results on the algebraic functions $X(y)$ and $Y(x)$ determined by equation (2.1) and their Riemann surface $\mathbf{S}$. Some of these have already been stated in Subsection 2.1.

The structure of the proofs of Theorems 2.1–2.6 is similar. We need some additional lemmas to describe it.

Let
$$D = \{x: \ |x| < 1\}, \qquad \Gamma = \partial D = \{x: \ |x| = 1\},$$
$D \subset \mathbf{C}$, $\Gamma \subset \mathbf{C}$, where $\mathbf{C}$ is the complex plane.

**Lemma 3.1.** *The algebraic function $Y(x)$ has two branches on $\Gamma$, denoted by $Y_0(x)$ and $Y_1(x)$, $Y_0(1) < Y_1(1)$.*

1. *If $\mathsf{E}_y > 0$, then $|Y_0(x)| < 1$ and $|Y_1(x)| \geq 1$. Only at the point $x = 1$ we have $|Y_1(x)| = 1$, in particular $Y_1(1) = 1$. Moreover, $Y_0(x)$ [resp. $Y_1(x)$] is a real analytic curve on $\Gamma$ contained in [resp. out] the unit circle $\Gamma$ for $x \neq 1$.*

2. *If $\mathsf{E}_y < 0$, then $|Y_0(x)| \leq 1$ and $|Y_1(x)| > 1$. Only at the point $x = 1$ we have $|Y_0(x)| = 1$, in particular $Y_0(1) = 1$. Moreover, $Y_0(x)$ [resp. $Y_1(x)$] is a real analytic curve on $\Gamma$ contained in [resp. out] the unit circle $\Gamma$ for $x \neq 1$.*

*Similar properties hold for the algebraic function $X(y)$, which has two branches $X_0(y)$ and $X_1(y)$.*

*Proof.* See [9]. □

Define the following sets on the Riemann surface $\mathbf{S}$:

$$\begin{aligned} \Gamma_0 &= h_x^{-1}(\Gamma) \cap \{s: \ |y(s)| \leq 1\}, \qquad & \Gamma_1 &= h_x^{-1}(\Gamma) \cap \{s: \ |y(s)| \geq 1\}; \\ \widetilde{\Gamma}_0 &= h_y^{-1}(\Gamma) \cap \{s: \ |x(s)| \leq 1\}, \qquad & \widetilde{\Gamma}_1 &= h_y^{-1}(\Gamma) \cap \{s: \ |x(s)| \geq 1\}. \end{aligned} \tag{3.1}$$

**Lemma 3.2.** *The sets $\Gamma_0, \Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ are closed analytic curves without self-intersections. They belong to the same homology class, which is one of the normal homology bases on the torus.*

1. *If $\mathsf{E}_x > 0, \mathsf{E}_y > 0$, then $\Gamma_0 \subset h_y^{-1}(D)$, $\widetilde{\Gamma}_0 \subset h_x^{-1}(D)$, $\Gamma_1 \cap h_y^{-1}(\overline{D}) = s_E$, $\widetilde{\Gamma}_1 \cap h_x^{-1}(\overline{D}) = s_E$ and $h_x^{-1}(\Gamma) \cap h_y^{-1}(\Gamma) = \Gamma_1 \cap \widetilde{\Gamma}_1 = s_E$, where $x(s_E) = y(s_E) = 1$, see Figure 3.1(a).*

2. *If $\mathsf{E}_x < 0, \mathsf{E}_y < 0$, then $\Gamma_0 \subset h_y^{-1}(\overline{D})$, $\widetilde{\Gamma}_0 \subset h_x^{-1}(\overline{D})$, $\Gamma_1 \cap h_y^{-1}(\overline{D}) = \emptyset$, $\widetilde{\Gamma}_1 \cap h_x^{-1}(\overline{D}) = \emptyset$ and $h_x^{-1}(\Gamma) \cap h_y^{-1}(\Gamma) = \Gamma_0 \cap \widetilde{\Gamma}_0 = s_E$, where $x(s_E) = y(s_E) = 1$, see Figure 3.1(b).*

*Proof.* See [9]. □

In particular,

- if $\mathsf{E}_x > 0, \mathsf{E}_y > 0$, then $\Gamma_1 \cap F_0 = \widetilde{\Gamma}_1 \cap F_0 = (1,1) = s_E$, $\Gamma_0 \cap F_0 = (1, p_{0-1}/p_{01})$ and $\widetilde{\Gamma}_0 \cap F_0 = (p_{-10}/p_{10}, 1)$;
- if $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$, then $\Gamma_0 \cap F_0 = \widetilde{G}_0 \cap F_0 = (1,1) = s_E$, $\Gamma_1 \cap F_0 = (1, p_{0-1}/p_{01}) = s_E^*$ and $\widetilde{\Gamma}_1 \cap F_0 = (p_{-10}/p_{10}, 1) := \widetilde{s}_E^*$.

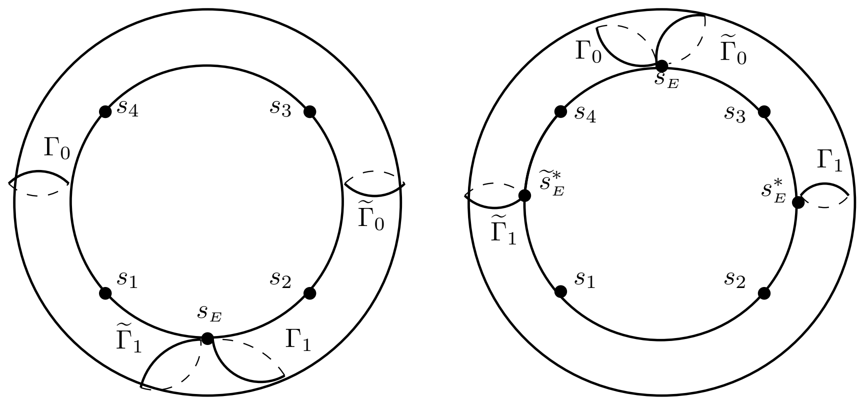


Figure 3.1(a). $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$. Figure 3.1(b). $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$.

We orient $\Gamma_0$ in such a way, that rotation along $\Gamma_0$ implies positive rotation along $\Gamma = \{x : \ |x| = 1\}$ on $\mathbf{C}$. The curves $\Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ are oriented homologically to $\Gamma_0$. It follows that rotation along $\Gamma_1$ implies negative rotation along $\Gamma$ on $\mathbf{C}$.

Introduce also the following differential form on $\mathbf{S}$:

$$d\omega = \frac{dx}{2a(x)y + b(x)} = -\frac{dy}{2\widetilde{a}(y)x + \widetilde{b}(y)}, \tag{3.2}$$

where

$$Q(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(x)y + \widetilde{c}(x). \tag{3.3}$$

*Structure of the proofs of Theorems* 2.1–2.6.

To find the Martin boundary, it is sufficient to find the asymptotics of the Green function $\pi_{ij}^{i_0 j_0}$ for $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ as $r \to \infty$, $\gamma(r) \to \gamma$. Then it remains to use the definition of the Martin kernel $\mathsf{k}_{ij}(i_0, j_0) = \pi_{ij}^{i_0 j_0}/\pi_{ij}^{00}$ (so that the reference measure is the Dirac measure at the point $(0,0)$).

First of all, we derive a functional equation for the generating functions of $\pi_{ij}^{i_0 j_0}$, see (3.7), (3.35), (3.75). In the quarter plane, the functional equation is quite similar to the equation for the stationary probabilities in case of ergodicity. This has been thoroughly analysed in [8].

Using Cauchy's theorem, $\pi_{ij}^{i_0j_0}$ can be represented as a double integral, cf. (3.15), (3.40) or (3.100). Using two-dimensional residues, this double integral is transformed into a one-dimensional integral over a Riemann surface of genus 1. Some space is required for gathering the necessary information on the corresponding elliptic curve and especially on the real points of this curve.

The integrand contains the unknown functions $\pi^{i_0j_0}(s)$ and $\widetilde{\pi}^{i_0j_0}(s)$. All we need from these functions, is their singularities. A priori, these functions are defined in some domains on $\mathbf{S}$ as $\pi^{i_0j_0}(s) := \pi^{i_0j_0}(x(s))$, $\widetilde{\pi}^{i_0j_0}(s) := \pi^{i_0j_0}(y(s))$. The crucial property is that they can be meromorphically continued on $\mathbf{S}$. This is carefully described in each case.

The integrals on $\mathbf{S}$ are typical examples for applying the saddle-point method, and moreover, nice analyticity properties allow us to deform the integration contour along the Riemann surface.

When deforming this contour, we may encounter the poles of the functions in the integrand $\pi^{i_0j_0}(s)$, $\widetilde{\pi}^{i_0j_0}(s)$. If this is the case, the asymptotics of $\pi_{ij}^{i_0j_0}$ is determined by the "lowest" of these poles, which will be always on $F_0$; otherwise the asymptotics of $\pi_{ij}^{i_0j_0}$ will be determined by the contribution of the saddle-point $s(\gamma)$.

Note also that the poles of the integrands occur at the points $s$, such that $q(x(s), p_{0-1}/(p_{01}y(s))) = 0$ or $\widetilde{q}(p_{-10}/(p_{10}x(s)), y(s)) = 0$. (In particular the points $s(\gamma')$, $s(\gamma'')$, $s(\gamma_E^*)$, where $\gamma'$, $\gamma''$, $\gamma_E^*$ are defined in Subsections 2.3, 2.4 and 2.5, are exactly the poles.)

The main contribution to the Martin boundary comes from the saddle-points: taking different $\gamma$ in such a way that the asymptotics of $\pi_{ij}^{i_0j_0}$ is determined by the saddle-point, we will obtain different points of the Martin boundary as e.g. in (2.6), (2.21), (2.29), (2.39), (2.40). On the contrary, the poles do not contribute much: the angles $\gamma$ such that the asymptotics of $\pi_{ij}^{i_0j_0}$ is determined by a given pole, will add only one point to this boundary, as e.g. in (2.22), (2.23), (2.28), (2.30), (2.41), (2.42).

Next, we prove our statements on the Riemann surface $\mathbf{S}$. We will need all of them, when showing Theorems 2.1–2.6.

*Proof of Lemma* 2.1. The equation $Q(x, y) = 0$ can be represented in the form

$$Q(x, y) = a(x)y^2 + b(x)y + c(x) = 0$$

with the discriminant

$$\begin{aligned} D(x) &= b^2(x) - 4(x)c(x) \\ &= p_{10}^2 x^4 - 2p_{10}x^3 + (1 + 2p_{10}p_{0-1} - 4p_{01}p_{-10})x^2 - 2p_{0-1}x + p_{0-1}^2. \end{aligned}$$

The branch points of $Y(x)$ are the zeros of $D(x)$. (The analogous arguments are true for $X(y)$.) Then these branch points can be found explicitly:

$$x_{1,2} = \Big(1 \pm 2\sqrt{p_{01}p_{0-1}} - \sqrt{1 \pm 4\sqrt{p_{01}p_{0-1}} + 4p_{01}p_{0-1} - 4p_{10}p_{-10}}\Big)/2p_{10},$$

$$x_{3,4} = \left(1 \pm 2\sqrt{p_{01}p_{0-1}} + \sqrt{1 \pm 4\sqrt{p_{01}p_{0-1}} + 4p_{01}p_{0-1} - 4p_{10}p_{-10}}\right)/2p_{10},$$

$$y_{1,2} = \left(1 \pm 2\sqrt{p_{10}p_{-10}} - \sqrt{1 \pm 4\sqrt{p_{10}p_{-10}} + 4p_{10}p_{-10} - 4p_{01}p_{0-1}}\right)/2p_{01},$$

$$y_{3,4} = \left(1 \pm 2\sqrt{p_{10}p_{-10}} + \sqrt{1 \pm 4\sqrt{p_{10}p_{-10}} + 4p_{10}p_{-10} - 4p_{01}p_{0-1}}\right)/2p_{01}.$$

□

*Proof of Lemma* 2.2. This is a corollary of the previous lemma. The discriminant $D(x)$ being a polynomial of degree four without multiple zeros, the Riemann surface of $X(y)$ is homeomorphic to the torus. The same holds for $X(y)$. □

*Proof of Lemma* 2.3. In the neighbourhood of any $s \in \mathbf{S}$, one of the functions $x, y, 1/x, 1/y$ will act as the uniformisation variable. Assume that it is e.g. $x$, and that $s \in S_r$. Then in a small neighbourhood of $s$, the set of the points of $S_r$ forms an analytic arc. It follows that $S_r$ is an analytic curve without self-intersections. Moreover, $S_r$ being a closed set, all its components are closed.

Let us recall now that the values of $Y(x)$ are real for $x_2 \le x \le x_3$, since $Y(1)$ is real. But $Y(x_1)$ and $Y(x_4)$ are also real. Thus $Y(x)$ is not real for $x_1 < x < x_2$ and $x_3 < x < x_4$ and real for $x < x_1$, $x > x_4$. So, there are two components of $S_r$ by construction of the Riemann surface.

We denote by $F_0$ the component of $S_r$, where $x_2 \le x(s) \le x_3$, and by $F_1$ the other one. Let us note that for $s \in F_0$ also $y_2 \le y(s) \le y_3$. If $s \in F_1$ and $0 < y(s) \le y_1$ or $y(s) \ge y_4$, then $x(s) < 0$; if $s \in F_1$ and $y(s) < 0$, then $0 < x(s) \le x_1$ or $y(s) \ge y_4$. If $y(s) = 0$ then $x(s) = 0$ or $\infty$; and if $x(s) = 0$, then $y(s) = 0$ or $\infty$. □

*Proof of Lemma* 2.4. We prove this lemma for $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$. The other case is similar.

Let $\gamma = 0$. Then $\chi_0(s) = |x(s)|$ has four critical points $s_i(0)$, $i = 1, 2, 3, 4$, such that

$$x_i(0) = x_i \qquad \text{for } i = 1, 2, 3, 4;$$

$$y_i(0) = \sqrt{p_{0-1}/p_{01}} \quad \text{for } i = 2, 3; \qquad y_i(0) = -\sqrt{p_{0-1}/p_{01}} \quad \text{for } i = 1, 4;$$

$$\chi_0(s_1(0)) < \chi_0(s_2(0)) < \chi_0(s_3(0)) < \chi_0(s_4(0)),$$

$$s_2(0) = s_3, \qquad s_3(0) = s_1.$$

Let now $0 < \gamma < \pi/2$. Alternative equations for determining the critical points are:

$$\left(xy^{\operatorname{tg}\gamma}\right)'_x = y^{\operatorname{tg}\gamma - 1}\left(y + \operatorname{tg}\gamma\, x\frac{dy}{dx}\right) = 0,$$

$$\left(xy^{\operatorname{tg}\gamma}\right)'_y = y^{\operatorname{tg}\gamma - 1}\left(y\frac{dy}{dx} + \operatorname{tg}\gamma\, x\right) = 0.$$

They are reduced to

$$\frac{y}{\operatorname{tg}\gamma\, x} = -\frac{dy}{dx} = \frac{p_{10} - p_{-10}/x^2}{p_{01} - p_{0-1}/y^2}. \tag{3.4}$$

This equation together with $Q(x,y) = 0$ gives the system

$$\begin{cases} p_{10}(1 + \operatorname{tg}\gamma)x + \dfrac{p_{-10}}{x}(1 - \operatorname{tg}\gamma) - 1 & = & -\dfrac{2p_{0-1}}{y} \\ p_{10}(1 - \operatorname{tg}\gamma)x + \dfrac{p_{-10}}{x}(1 + \operatorname{tg}\gamma) - 1 & = & -2p_{0-1}y. \end{cases} \tag{3.5}$$

If $\gamma \neq \pi/4$, this system has four roots $(x_i(\gamma), y_i(\gamma))$, $i = 1,2,3,4$. They depend continuously on $\gamma$. So, we have four critical points $s_i(\gamma)$ on $\mathbf{S}$. Moreover, $s_2(\gamma), s_3(\gamma) \in F_0$, $s_1(\gamma), s_4(\gamma) \in F_1$ since this holds for $\gamma = 0$. If $\gamma = \pi/4$, the system (3.5) has two roots. The corresponding critical points $s_2(\pi/4), s_3(\pi/4)$ are on $F_0$. (One can also obtain from (3.4) and $Q(x,y) = 0$ two points $s_1(\pi/4) = (0,0)$, $s_4(\pi/4) = (\infty,\infty)$ on $F_1$, but they are not on $\chi^{-1}_{\pi/4}(0,\infty)$.)

Let us show that $s_2(\gamma) \in [s_3, s_4)$ for $0 \leq \gamma < \pi/2$. Note that $y_2(\gamma) = \sqrt{p_{0-1}/p_{01}}$ only for $\gamma = 0$. In fact, substituting $y = \sqrt{p_{0-1}/p_{01}}$ into (3.4), we get $x = \pm\sqrt{p_{-10}/p_{10}}$ for $\gamma \neq 0$. But because of the equation $Q(x,y) = 0$ it is impossible that simultaneously $x = \pm\sqrt{p_{-10}/p_{10}}$, $y = \pm\sqrt{p_{-10}/p_{10}}$. Thus, $s_3$ is a critical point only for $\gamma = 0$. Similarly $x_2(\gamma) \neq \sqrt{p_{0-1}/p_{01}}$, so $s_2, s_4$ can not be critical points for any $0 \leq \gamma < \pi/2$. Since $s_2(\gamma)$ depends continuously on $\gamma$ and taking into account the above, we conclude that only one of the following cases can occur: $s_2(\gamma) \in [s_3, s_4)$ or $s_2(\gamma) \in [s_2, s_3)$ for all $0 \leq \gamma < \pi/2$. To reject the second case, it is sufficient to show that $y_2(\gamma) < y_2(0) = \sqrt{p_{0-1}/p_{01}}$. This is easily seen from (3.4). The left-hand side in (3.4) is positive. The numerator in the right-hand side is negative, since $x(s) < x(s_2) = x(s_4) = \sqrt{p_{-10}/p_{10}}$ for all $s \in (s_2, s_4)$. Then the denominator should be negative too, thus $p_{01} - p_{0-1}/y^2(\gamma) < 0$.

One can prove by the same way that $s_3(\gamma) \in [s_1, s_2)$ for all $0 \leq \gamma < \pi/2$.

The above implies that for all $0 < \gamma < \pi/2$

$$x_2(0) < x_2(\gamma) < x(s_4) = x(s_2) < x_3(\gamma) < x_3(0),$$

$$y_2(\gamma) < y_2(0) = y_(s_1) = y_(s_3) < y_3(\gamma),$$

hence $\chi_2(\gamma) < \chi_3(\gamma)$. In the same way, one can study the "real circle" $F_1$ and deduce that $\chi_1(\gamma) < \chi_2(\gamma)$, $\chi_3(\gamma) < \chi_4(\gamma)$. These last facts imply in particular, that $s_2(\gamma) \neq s_3(\gamma)$ and $s_1(\gamma) \neq s_4(\gamma)$ can not occur for any $0 \leq \gamma < \pi/2$. Non-degeneracy follows.

Next, we will show that $s_2(\gamma_1) \neq s_2(\gamma_2)$ and $s_3(\gamma_1) \neq s_3(\gamma_2)$ for all $0 < \gamma_1 < \gamma_2 < \pi/2$. Let us suppose e.g. that $x_2(\gamma_1) = x_2(\gamma_2)$ for some $\gamma_1, \gamma_2$. Recall that $0 < x_2(\gamma_1) < \sqrt{p_{-10}/p_{10}}$ for $s(\gamma_1) \in (s_3, s_4)$. Then by (3.5)

$$-2p_{01}y_2(\gamma_1) \quad = \quad p_{10}(1 - \operatorname{tg}\gamma_1)x_2(\gamma_1) + \frac{p_{-10}}{x_2(\gamma_1)}(1 + \operatorname{tg}\gamma_1) - 1$$

$$\begin{aligned}
&= p_{10}(1-\operatorname{tg}\gamma_1)x_2(\gamma_2) + \frac{p_{-10}}{x_2(\gamma_2)}(1+\operatorname{tg}\gamma_1) - 1 \\
&< p_{10}(1-\operatorname{tg}\gamma_1)x_2(\gamma_1) + \frac{p_{-10}}{x_2(\gamma_1)}(1+\operatorname{tg}\gamma_1) - 1 \\
&= -2p_{01}y_2(\gamma_2).
\end{aligned}$$

Let $\gamma = \pi/2$. The function $\chi_\gamma(s) = |y(s)|$ has four critical points $s_i(\pi/2)$, $s_2(\pi/2) = s_4$, $s_3(\pi/2) = s_2$; $y_i(\pi/2) = y_i$, $i = 1,2,3,4$.

It remains to prove, that $\lim_{\gamma\to\pi/2} s_i(\gamma) = s_i(\pi/2)$. To this end introduce the function $\chi'_\gamma(s) = |x^{\operatorname{ctg}\gamma}(s)y(s)|$, where $0 < \gamma \leq \pi/2$. It has the critical points ${s'}_i(\gamma)$, $i = 1,2,3,4$. One can study these similarly $s_i(\gamma)$ for $\chi_\gamma(s)$ and deduce that $\lim_{\gamma\to\pi/2} s'_i(\gamma) = s'_i(\pi/2)$, since $\lim_{\gamma\to 0} s_i(\gamma) = s_i(0)$. Moreover, $s_i(\gamma) = s'_i(\gamma)$ for $\gamma < 0 \leq \pi/2$ by the definition of the critical points. Then

$$\lim_{\gamma\to\pi/2} s_i(\gamma) = \lim_{\gamma\to\pi/2} {s'}_i(\gamma) = {s'}_i(\pi/2) = s_i(\pi/2).$$

The case $\gamma \in (\pi/2, \pi]$ is quite similar. □

*Remark* 3.1. The function $\chi'_\gamma(s) = |x^{\operatorname{ctg}\gamma}(s)y(s)|$, $0 \leq \gamma < \pi$ (for $\gamma = 0$, put $\chi'_0(s) = |x(s)|$), has the critical points $s'_i(\gamma)$, $i = 1,2,3,4$, with the same properties as $s_i(\gamma)$ for $\chi_\gamma(s)$. Moreover, $s'_i(\gamma) = s_i(\gamma)$ for all $0 \leq \gamma < \pi$.

*Remark* 3.2. By the maximum modulus principle all these critical points have index 1 in the sense of Morse theory [11]. Then the level curves $\{s : \chi_\gamma(s) = \chi_\gamma(s_i)\}$ are orthogonal in these points and they subdivide their sufficiently small neighbourhoods into four sections.

Next, we have to analyse the level curves $\{s : \chi_\gamma(s) = c\}$ of the function $\chi_\gamma(s)$. When $\gamma = 0$, we have $\chi_0(s) = |x(s)|$, and the way they look like, is easily seen from the construction of the Riemann surface. The following lemma shows that for $\gamma > 0$ there are no bifurcations. This property is usually called *structural stability* of level curves.

**Lemma 3.3.** *For any $\gamma_1$, $\gamma_2$, $0 \leq \gamma_1, \gamma_2 < \pi$, $\gamma_1, \gamma_2 \neq \pi/4, 3\pi/4$ there exist homeomorphisms $f_{\gamma_1\gamma_2} : \mathbf{S} \to \mathbf{S}$ and $g_{\gamma_1,\gamma_2} : [0,\infty] \to [0,\infty)$, such that the diagram*

$$\begin{array}{rcl}
\mathbf{S} & \xrightarrow{\chi_{\gamma_1}} & [0,\infty] \\
f_{\gamma_1,\gamma_2}\downarrow & & \downarrow g_{\gamma_1,\gamma_2} \\
\mathbf{S} & \xrightarrow{\chi_{\gamma_2}} & [0,\infty]
\end{array}$$

*is commutative.*

*Proof.* The proof is similar to the proof of Theorem 1 from [1] if we take into account Lemma 2.4. The difference is the following. Instead of the function $\chi_\gamma(s)$, it is convenient to consider the function $\widetilde{\chi}_\gamma(s) = 2\pi^{-1}\operatorname{Arctg}\chi_\gamma(s) : \mathbf{S} \to [0,1]$. It is not differentiable in the points $\widetilde{\chi}^{-1}(\{0,1\})$, but this can be corrected by smoothing. We get new critical points that are conserved w.r.t. perturbation. This establishes the proof. □

**Corollary 3.1.** *For any $\gamma$, $0 \le \gamma < \pi$, the set $D_\chi = \{s : \ \chi_\gamma(s) < \chi\}$ is homeomorphic to $h_x^{-1}(D) = \{s : \ |x(s)| < 1\}$, where $\chi = \chi_\gamma(s_3(\gamma)) - \varepsilon$ for all sufficiently small $\varepsilon > 0$. The set $\overline{D}_\chi$ is homeomorphic to $h_x^{-1}(\overline{D})$ after identifying the points $(1,1)$ and $(1, p_{0-1}/p_{01})$. Moreover, under this isomorphism these identified points are mapped to the point $s_3(\gamma)$.*

We will also use the construction of the *Galois automorphisms* on $\mathbf{S}$:

$$\xi : \mathbf{S} \to \mathbf{S}, \qquad \eta : \mathbf{S} \to \mathbf{S}.$$

It is given in detail in [9]. We will only mention that

$$\begin{array}{ll} s' = \xi s & \text{if } x(s') = x(s); \\ s'' = \eta s & \text{if } y(s'') = y(s). \end{array} \tag{3.6}$$

This implies the assertions:

$$x(\xi s) = x(s), \qquad y(\xi s) = \frac{p_{0-1}}{p_{01} x(s)};$$

$$y(\eta s) = y(s), \qquad x(\eta s) = \frac{p_{-10}}{p_{10} y(s)}.$$

Obviously, the points $s_1$, $s_3$ [resp. $s_2$, $s_4$] are the fixed points of $\xi$ [resp. $\eta$] and

$$\xi s_2 = s_4, \qquad \eta s_1 = s_3;$$

$$\xi^2 = Id, \qquad \eta^2 = Id.$$

Finally, let us give some ubiquitous notations and definitions. We denote by $P_{ij}^{i_0 j_0}(t)$ the probability of being at point $(i,j)$ at time $t$, when the initial state is $(i_0, j_0)$. Introduce the generating functions

$$\pi_{ij}^{i_0 j_0}(z) = \sum_{t=0}^{\infty} P_{ij}^{i_0 j_0}(t) z^t.$$

Note that $\pi_{ij}^{i_0 j_0}(1)$ is finite, since it is the mean number of visits to state $(i,j)$ starting at $(i_0, j_0)$. So $\pi_{ij}^{i_0 j_0}(z) < \infty$, for $|z| \le 1$. In the notation of Section 2, $\pi_{ij}^{i_0 j_0}(1) = \pi_{ij}^{i_0 j_0}$.

The following functions on the Riemann surface are defined as

$$\begin{array}{rcll} f_*^{i_0 j_0}(s) & := & f_*^{i_0 j_0}(x(s), y(s)), & \\ q(s) & := & q(x(s), y(s)), & \\ \widetilde{q}(s) & := & \widetilde{q}(x(s), y(s)), & \\ q_0(s) & := & q_0(x(s), y(s)), & \qquad s \in \mathbf{S}. \end{array}$$

### 3.2. Random walk in $\mathbf{Z}^2$: proofs

We restrict ourselves to the case $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$. The proof is similar for the other cases.

**Lemma 3.4.** *If $|x| = 1$, $|y| = 1$, $|z| < 1$, the following equation holds*

$$Q(x,y,z) \sum_{(i,j)\in S} \pi_{ij}^{i_0 j_0}(z) x^{i-1} y^{j-1} = x^{i_0} y^{j_0} + f_*^{i_0 j_0}(x,y,z), \tag{3.7}$$

*where*

$$\begin{aligned} Q(x,y,z) &= xy\Big(1 - z\sum_{i,j} p_{ij} x^i y^j\Big), \\ f_*^{i_0 j_0}(x,y,z) &= \sum_{m=1}^{n} q_m(x,y,z) \sum_{(i,j)\in S^m} \pi_{ij}^{i_0 j_0}(z) x^i y^j, \\ q_m(x,y,z) &= z\sum_{i,j} {}^{(m)}p_{ij} x^i y^j - 1, \qquad m = 1,\dots,n. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} P_{ij}^{i_0 j_0}(t+1) &= \sum_{(k,l)\in S} p_{i-k,j-l} P_{kl}^{i_0 j_0}(t) \\ &\quad + \sum_{m=1}^{n} \sum_{(k,l)\in S^m} {}^{(m)}p_{i-k,j-l} P_{kl}^{i_0 j_0}(t). \end{aligned} \tag{3.8}$$

This yields

$$\begin{aligned} &\pi_{ij}^{i_0 j_0}(z) - \pi_{ij}^{i_0 j_0}(0) \\ &\quad = z\Big(\sum_{(k,l)\in S} p_{i-k,j-l}\pi_{kl}^{i_0 j_0}(z) + \sum_{m=1}^{n}\sum_{(k,l)\in S^m} {}^{(m)}p_{i-k,j-l}\pi_{kl}^{i_0 j_0}(z)\Big). \end{aligned} \tag{3.9}$$

If $z = 0$, then $\pi_{i_0 j_0}^{i_0 j_0}(0) = 1$ and $\pi_{ij}^{i_0 j_0}(0) = 0$ for $(i,j) \neq (i_0, j_0)$. Let $|z| < 1$. Multiplying equation (3.9) by $x^i y^j$, where $|x|, |y| = 1$, taking the summation over $i, j$, and changing the order of summation, we get

$$\begin{aligned} &\sum_{(i,j)\in S} \pi_{ij}^{i_0 j_0}(z) x^i y^j + \sum_{m=1}^{n}\sum_{(i,j)\in S^m} \pi_{ij}^{i_0 j_0}(z) x^i y^j - x^{i_0} y^{j_0} \\ &\quad = zp(x,y) \sum_{(i,j)\in S} \pi_{ij}^{i_0 j_0}(z) x^i y^j + z\sum_{m=1}^{n} p_m(x,y) \sum_{(i,j)\in S^m} \pi_{ij}^{i_0 j_0}(z) x^i y^j, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} p(x,y) &= \sum_{i,j} p_{ij} x^i y^j, \\ p_m(x,y) &= \sum_{i,j} {}^{(m)}p_{ij} x^i y^j, \qquad m = 1,\dots,n. \end{aligned}$$

The sum over $\{(i,j) \in S\}$ in the left-hand side of (3.10) is finite:

$$\sum_{(i,j)} \pi_{ij}^{i_0 j_0}(z)|x|^i|y|^j = \sum_{t=0}^{\infty} \sum_{(i,j)} P_{ij}^{i_0 j_0}(t)|z|^t = \sum_{t=0}^{\infty} |z|^t < \infty.$$

Thus equation (3.7) holds. □

Recall that we are interested in the asymptotics of $\pi_{ij}^{i_0 j_0} = \pi_{ij}^{i_0, j_0}(1)$. For $z=1$ in accordance with notation (2.1), (2.4), (2.5) we have

$$\begin{aligned} Q(x,y,1) &= Q(x,y), \\ q_m(x,y,1) &= q_m(x,y), \qquad m = 1, \dots, n, \\ f_*^{i_0 j_0}(x,y,1) &= f_*^{i_0 j_0}(x,y). \end{aligned}$$

Introduce also the functions $a(x,z)$, $b(x,z)$, $c(x,z)$, $\widetilde{a}(x,z)$, $\widetilde{b}(x,z)$, $\widetilde{c}(x,z)$ by

$$Q(x,y,z) = a(x,z)y^2 + b(x,z)y + c(x,z) = \widetilde{a}(y,z)x^2 + \widetilde{b}(y,z)x + \widetilde{c}(x,z).$$

In accordance with (3.3) for $z=1$, we have

$$a(x,1) = a(x), \quad b(x,1) = b(x), \quad c(x,1) = c(x),$$

$$\widetilde{a}(x,1) = \widetilde{a}(x), \quad \widetilde{b}(x,1) = \widetilde{b}(x), \quad \widetilde{c}(x,1) = \widetilde{c}(x).$$

**Lemma 3.5.** *For all sufficiently large $j > 0$ and all $i \in \mathbf{Z}$*

$$\pi_{ij}^{i_0 j_0} = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{x^{i_0}(s) y^{j_0}(s) + f_*^{i_0 j_0}(s)}{x^i(s) y^j(s)}\, d\omega; \tag{3.11}$$

*for all sufficiently large $j < 0$ and all $i \in \mathbf{Z}$*

$$\pi_{ij}^{i_0 j_0} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{x^{i_0}(s) y^{j_0}(s) + f_*^{i_0 j_0}(s)}{x^i(s) y^j(s)}\, d\omega; \tag{3.12}$$

*for all sufficiently large $i > 0$ and all $j \in \mathbf{Z}$*

$$\pi_{ij}^{i_0 j_0} = \frac{1}{2\pi i} \int_{\widetilde{\Gamma}_1} \frac{x^{i_0}(s) y^{j_0}(s) + f_*^{i_0 j_0}(s)}{x^i(s) y^j(s)}\, d\omega; \tag{3.13}$$

*for all sufficiently large $i < 0$ and all $j \in \mathbf{Z}$*

$$\pi_{ij}^{i_0 j_0} = \frac{1}{2\pi i} \int_{\widetilde{\Gamma}_0} \frac{x^{i_0}(s) y^{j_0}(s) + f_*^{i_0 j_0}(s)}{x^i(s) y^j(s)}\, d\omega; \tag{3.14}$$

*where the differential form $d\omega$ and the curves $\Gamma_0$, $\Gamma_1$, $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$ are defined by* (3.1) *and* (3.2).

*Proof.* For any $z = 1 - \varepsilon$ ($\varepsilon > 0$) fixed, equation (3.7) implies

$$\pi_{ij}^{i_0 j_0}(1-\varepsilon) = \frac{1}{(2\pi i)^2} \int\limits_{|x|=1} \int\limits_{|y|=1} \frac{x^{i_0} y^{j_0} + f_*^{i_0 j_0}(x,y,1-\varepsilon)}{x^i y^j Q(x,y,1-\varepsilon)} \, dy \, dx. \tag{3.15}$$

We will show (3.11) and (3.12).

Let us fix $x$ with $|x| = 1$. The inner integral in (3.15)

$$\frac{1}{2\pi i} \int\limits_{|y|=1} \frac{x^{i_0} y^{j_0} + f_*^{i_0 j_0}(x,y,1-\varepsilon)}{x^i y^j Q(x,y,1-\varepsilon)} \, dy \tag{3.16}$$

equals the sum of the residues at the poles of the integrand inside or outside the circle $|y| = 1$ with "+" or "−" signs respectively. Whenever $|x| = 1$, the function $Q(x,y,1-\varepsilon)$ of $y$ has two zeros $Y_0(x,1-\varepsilon)$, $Y_1(x,1-\varepsilon)$, which are such, that $|Y_0(x,1-\varepsilon)| < 1$, $|Y_1(x,1+\varepsilon)| > 1$. (If $x \neq 1$ this is ensured by Lemma 3.1, if $x = 1$ this is easily shown explicitly.) Then the poles of the integrand

$$\frac{x^{i_0} y^{j_0} + f_*^{i_0 j_0}(x,y,1-\varepsilon)}{x^i y^j Q(x,y,1-\varepsilon)}$$

can occur only at the points $y = Y_0(x,1-\varepsilon), Y_1(x,1-\varepsilon), 0, \infty$. The residue at $y = 0$ is zero for all sufficiently large $j < 0$, since $S_1, \dots, S_n$ are finite. It can be non-zero for all $j > 0$. The residue at $y = \infty$ is zero for all sufficiently large $j > 0$ and can be non-zero for $j < 0$. Thus the integral (3.16) equals the residue of the integrand at $Y_1(x,1-\varepsilon)$ with "−" sign for $j > 0$ and it equals the residue at $Y_0(x,1-\varepsilon)$ for $j < 0$. Hence, for sufficiently large $j > 0$

$$\pi_{ij}^{i_0 j_0}(z) = -\frac{1}{2\pi i} \int\limits_{|x|=1} \frac{x^{i_0} Y_1^{j_0}(x,z) + f_*^{i_0 j_0}(x, Y_1(x,z), z)}{x^i Y_1^j(x,z)\big(2a(x,z)Y_1(x,z) + b(x,z)\big)} \, dx$$

and for sufficiently large $j < 0$

$$\pi_{ij}^{i_0 j_0}(z) = \frac{1}{2\pi i} \int\limits_{|x|=1} \frac{x^{i_0} Y_0^{j_0}(x,z) + f_*^{i_0 j_0}(x, Y_0(x,z), z)}{x^i Y_0^j(x,z)\big(2a(x,z)Y_0(x,z) + b(x,z)\big)} \, dx,$$

where $z = 1 - \varepsilon$. Finally, let $z \to 1$ and recall the definitions of the curves $\Gamma_0$, $\Gamma_1$ (3.1), and of the form $d\omega$ (3.2). The representations (3.13) and (3.14) are obtained similarly by exchanging the roles of $x$ and $y$. □

**Lemma 3.6.** *Let* $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$, *and let* $(\gamma(r)) \to \gamma$ *as* $r \to \infty$, *where* $0 \le \gamma < 2\pi$. *If*

$$x^{i_0}(\gamma) y^{j_0}(\gamma) + f_*^{i_0 j_0}(x(\gamma), y(\gamma)) \neq 0, \tag{3.17}$$

*then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{C(\gamma, i_0, j_0)}{\sqrt{j}\, x^i(\gamma(r)) y^j(\gamma(r))}, \qquad \textit{for } \gamma \neq 0, \pi; \tag{3.18}$$

$$\pi_{ij}^{i_0 j_0} \sim \frac{\widetilde{C}(\gamma, i_0, j_0)}{\sqrt{i}\, x^i(\gamma(r)) y^j(\gamma(r))}, \qquad \textit{for } \gamma \neq \pi/2, 3\pi/2. \tag{3.19}$$

*Here*

$$C(\gamma, i_0, j_0) = \left[x^{i_0}(\gamma) y^{j_0}(\gamma) + f_*^{i_0 j_0}(x(\gamma), y(\gamma))\right]$$
$$\times \left|x^{\operatorname{ctg}\gamma}(\gamma) y(\gamma)\right|^{1/2} \left[2a(x(\gamma))y(\gamma) + b(x(\gamma))\right]^{-1} \left|\frac{d^2\, x^{\operatorname{ctg}\gamma}(\gamma) Y(x(\gamma))}{dx^2}\right|^{-1/2};$$
$$\widetilde{C}(\gamma, i_0, j_0) = \left[x^{i_0}(\gamma) y^{j_0}(\gamma) + f_*^{i_0 j_0}(x(\gamma), y(\gamma))\right]$$
$$\times \left|x(\gamma) y(\gamma)^{\operatorname{tg}\gamma}\right|^{1/2} \left[2\widetilde{a}(y(\gamma))x(\gamma) + \widetilde{b}(y(\gamma))\right]^{-1} \left|\frac{d^2\, X(y(\gamma)) y^{\operatorname{tg}\gamma}(\gamma)}{dy^2}\right|^{-1/2}$$

*and*

$$\sqrt{\operatorname{ctg}\gamma}\, C(\gamma, i_0, j_0) = \widetilde{C}(\gamma, i_0, j_0) \qquad \textit{for } \gamma \neq 0, \pi/2, \pi, 3\pi/2. \tag{3.20}$$

*Proof.* By virtue of Lemma 3.5 the mean number of visits to state $(i, j)$ can be written as an integral along one of the curves $\Gamma_1$, $\Gamma_0$, $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_0$. These integrals are typical for applying *the saddle-point method,* see [4].

Let us first look for the asymptotics of the integral

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{x^{i_0}(s) y^{j_0}(s) + f_*^{i_0 j_0}(x(s), y(s))}{\left(x^{\operatorname{ctg}\gamma}(s) y(s)\right)^j}\, d\omega, \tag{3.21}$$

as $j \to \infty$, where $\gamma \in (0, \pi)$. The point $s(\gamma)$ is a saddle-point. Our goal is to shift the integral contour to this point, avoiding singularities of the integrand and then use the saddle-point method.

If $\gamma < \gamma_E$, then $s(\gamma) \in (s_1, s_E)$; if $\gamma > \gamma_E$, then $s(\gamma) \in (s_E, s_3)$, where $s_E = \Gamma_1 \cap \widetilde{\Gamma}_1$. (Clearly, when $\gamma = \gamma_E$, there is no need to shift the contour.) The level curves $\{s : \chi'_\gamma(s) = \chi'_\gamma(s(\gamma))\}$ of the function $\chi'_\gamma(s) = |x^{\operatorname{ctg}\gamma}(s) y(s)|$ at $s(\gamma)$ are orthogonal and subdivide the neighbourhood of $s(\gamma)$ into four sectors. By structural stability (Theorem 3.3) they are homological to $\Gamma_1$ and intersect only at $s(\gamma)$, since this occurs for $\gamma = 0$.

In a sufficiently small neighbourhood $U$ of $s(\gamma)$, the curves of steepest descent $\{s : \operatorname{Im}\ln x^{\operatorname{ctg}\gamma}(s) y(s) = \operatorname{Im}\ln x(\gamma)^{\operatorname{ctg}\gamma} y(\gamma) = 0\}$ are orthogonal, see Lemma 1.3 Chapter IV in [4]. One of these is contained in $F_0$. Let the other be denoted by $\Gamma_u$. Introduce the closed curve $\Gamma_\gamma$ in $D_\gamma^+ = \{s : \chi_\gamma(s) > \chi_\gamma(s(\gamma))\}$ homological to $\Gamma_1$ and such that $\Gamma_\gamma \cap U = \Gamma_u$. Let $E_\gamma$ be a domain on $\mathbf{S}$ bounded by $\Gamma_1$, $\Gamma_\gamma$ and containing the interval $(s(\gamma), s_E)$ if $\gamma < \gamma_E$, and the interval $(s_E, s(\gamma))$ if $\gamma > \gamma_E$. The curve $\Gamma_\gamma$ can be chosen in such a way, that there are no poles of the integrand in $E_\gamma$, i.e. no points $s$, where $x(s)$ or $y(s)$ are zero or infinite. Due to Cauchy's theorem we may shift the integral contour to $\Gamma_\gamma$:

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{x^{i_0}(s)y^{j_0}(s)+f_*^{i_0j_0}(x(s),y(s))}{\left(x^{\operatorname{ctg}\gamma}(s)y(s)\right)^j}\,d\omega \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_\gamma} \frac{x^{i_0}(s)y^{j_0}(s)+f_*^{i_0j_0}(x(s),y(s))}{\left(x^{\operatorname{ctg}\gamma}(s)y(s)\right)^j}\,d\omega \\
&= \frac{1}{2\pi i}\int\limits_{h_x(\Gamma_\gamma)} \frac{x^{i_0}(s)Y^{j_0}(x)+f_*^{i_0j_0}(x(s),Y(x))}{\left(x^{\operatorname{ctg}\gamma}Y(x)\right)^j\left(2a(x)Y(x)+b(x)\right)}\,dx. \qquad (3.22)
\end{aligned}$$

By virtue of Theorem 1.7 in [4, Chapter IV], there exists a neighbourhood of $\gamma$, such that the asymptotics of the integral (3.22) is

$$\frac{1}{\left(x^{\operatorname{ctg}\gamma}(\gamma)y(\gamma)\right)^j}\Big(\sum_{k=0}^{n} c_k(\gamma)j^{-k-1/2}+o(j^{-k-1/2})\Big), \qquad (3.23)$$

as $j\to\infty$ uniformly in this neighbourhood. Moreover, $c_0(\gamma)=C(\gamma,i_0,j_0)$. Hence

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{x^{i_0}(s)y^{j_0}(s)+f_*^{i_0j_0}(x(s),y(s))}{\left(x^{\operatorname{ctg}\gamma}(s)y(s)\right)^j}\,d\omega \sim \frac{C(\gamma,i_0,j_0)}{\sqrt{j}\left(x^{\operatorname{ctg}\gamma}(\gamma)y(\gamma)\right)^j}$$

as $j\to\infty$ uniformly in the neighbourhood of $\gamma$. Then Lemma 3.5 together with the continuity of $C(\gamma,i_0,j_0)$ on $\gamma$ entails (3.18). To get (3.18) for $\pi<\gamma<2\pi$, and (3.19) for $-\pi/2<\gamma<\pi/2$ and $\pi/2<\gamma<3\pi/2$, we use (3.12), (3.13) and (3.14) respectively.

Note also that

$$\begin{aligned}
&\sqrt{\operatorname{ctg}\gamma}\,C(\gamma,i_0,j_0) \\
&= \sqrt{\operatorname{ctg}\gamma}\left[x^{i_0}(\gamma)y^{j_0}(\gamma)+f_*^{i_0j_0}(x(\gamma),y(\gamma))\right]\left|x(\gamma)^{\operatorname{ctg}\gamma}y(\gamma)\right|^{1/2} \\
&\quad\times\left[2\widetilde{a}(y(\gamma))X(y(\gamma))+\widetilde{b}(y(\gamma))\right]^{-1}\left|\frac{d^2\,x^{\operatorname{ctg}\gamma}(\gamma)Y(x(\gamma))}{dy^2}\right|^{-1/2} \\
&= \widetilde{C}(\gamma,i_0,j_0).
\end{aligned}$$

□

**Proposition 3.1.** *For all* $(i_0,j_0)$ *and all* $\gamma\in[0,2\pi)$

$$x^{i_0}(\gamma)y^{j_0}(\gamma)+f_*^{i_0j_0}(x(\gamma),y(\gamma))\neq 0. \qquad (3.24)$$

*Proof.* For all $\gamma\in[0,2\pi)$ there exists at least one pair $(i_0',j_0')$ satisfying (3.24). In fact, the function $f_*^{i_0,j_0}(x(\gamma),y(\gamma))$ is bounded on $\mathbf{Z}^2\times[0,2\pi]$, since

$$\pi_{ij}^{i_0j_0}\le\pi_{i_0j_0}^{i_0j_0}\le\sup_{(i_0,j_0)\in S^1\cup\cdots\cup S^n}\pi_{i_0j_0}^{i_0j_0},$$

while $x^{i_0}(\gamma)y^{j_0}(\gamma)$ can be made infinitely large by the choice of $(i_0, j_0)$, provided that $(x(\gamma), y(\gamma)) \neq (1,1)$. (If $x(\gamma) = 1$, $y(\gamma) = 1$ the left-hand side of (3.24) is always 1.)

Suppose that for some $(i_0'', j_0'')$ inequality (3.24) does not hold. Denote the mean number of visits to $(i,j)$ by $\pi'_{ij}$ and $\pi''_{ij}$, whenever the initial state of the chain is $(i_0', j_0')$ and $(i_0'', j_0'')$ respectively.

Let $\varepsilon_1$ and $\varepsilon_2$ be the probabilities of reaching $(i_0'', j_0'')$ and $(i_0', j_0')$ starting from $(i_0', j_0')$ and $(i_0'', j_0'')$ along some fixed path in $\mathbf{Z}^2$. Then for all sufficiently large $i, j$

$$\varepsilon_1 < \frac{\pi'_{ij}}{\pi''_{ij}} < \varepsilon_2. \tag{3.25}$$

If $i = r\cos(\gamma(r)), j = r\sin(\gamma(r))$ and $\gamma(r) \to \gamma$, then by (3.23)

$$\begin{aligned} x^i(\gamma(r))y^j(\gamma(r))\pi'_{ij} &\sim C(\gamma, i_0, j_0)j^{-1/2}, \\ x^i(\gamma(r))y^j(\gamma(r))\pi''_{ij} &= o(j^{-1/2}). \end{aligned}$$

Thus

$$\lim_{r\to\infty} \frac{\pi''_{ij}}{\pi'_{ij}} = 0,$$

which contradicts (3.25). □

*Proof of Theorem* 2.1. It follows immediately from Proposition 3.1, Lemma 3.6 and the definition of the Martin kernel that

$$\begin{aligned} \lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) &= \lim_{r\to\infty} \frac{\pi^{i_0 j_0}_{ij}}{\pi^{00}_{ij}} = \frac{C(\gamma, i_0, j_0)}{C(\gamma, 0, 0)} \\ &= \frac{x^{i_0}(\gamma)y^{j_0}(\gamma) + f_*^{i_0 j_0}(x(\gamma), y(\gamma))}{1 + f_*^{00}(x(\gamma), y(\gamma))}. \end{aligned} \tag{3.26}$$

By taking different $\gamma \in [0, 2\pi)$ in the right-hand side of (3.26), we get different non-negative harmonic functions of $(i_0, j_0)$. All of them are minimal. In fact, if this were not true, then one of these could be represented as an integral of the others by some finite measure. But this is not possible because of their asymptotics, whenever $i_0, j_0 \to \infty$. The proof of the theorem is concluded. □

### 3.3. Random walk in $\mathbf{Z}^+ \times \mathbf{Z}$, $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$: proofs

*Proof of Lemma* 2.5. The following statement is equivalent to our lemma: *the function $q(s)$ has a zero on the interval $((1, p_{0-1}/p_{01})\ ,\ s_1) \subset F_0$ if and only if $q(s_1) > 0$. The function $q(s)$ has a zero on the interval $(s_3\ ,\ (1, p_{0-1}/p_{01})) \subset F_0$ if and only if $q(s_3) > 0$. If $q(s_1) > 0$ [resp. $q(s_3) > 0$] this zero is unique.* Moreover, we will show that *on the corresponding interval the function $q(s)$ can not have zeros of multiplicity more than 1 for any parameters $\{p'_{ij}\}$.*

Let us consider the system of equations:

$$\begin{cases} q(x(s),y(s))=0 \\ q_x(s)=\dfrac{dY}{dx}(x(s))(p'_{11}x^2(s)+p'_{01}x(s)+p'_{-11}) \\ \qquad\qquad + (p'_{01}+2p'_{11}x(s))y(s)+2p'_{10}x(s)-1=0. \end{cases} \tag{3.27}$$

For given $\{p'_{ij}\}$ it determines the points, where $q(s)$ has zeros of multiplicity more than 1. Let us add to this system the equation

$$\sum_{i,j} p'_{ij}=1. \tag{3.28}$$

For any $s \in ((1, p_{0-1}/p_{01})\ ,\ s_1)$ [resp. $s \in (s_3\ ,\ (1, p_{0-1}/p_{01}))$], one can interpret (3.27)–(3.28) as a system of three linear equations with unknowns $p'_{ij}$. Suppose that for some $s$ belonging to the corresponding interval it has a solution $p'_{ij} \geq 0$. Let us move the point in question to $s_1$ [resp. to $s_3$]. Then $\dfrac{dY}{dx}(s) \to \infty$, since $x(s_1)=x_3$ [resp. $x(s_3)=x_2$] is a branch point for $Y(x)$. Hence, in view of the inequalities $0<x_2 \leq x(s) \leq x_3$, $0<y_2 \leq y(s) \leq y_3$, it follows from the second equation in (3.27) that there exists a "last" point $s_0$ where the system (3.27)–(3.28) has a solution $p'_{ij} \geq 0$. By dimensional considerations only two parameters of this solution may be different from zero. Indeed, suppose that at this point e.g. $p'_{-11}(s_0)>0$, $p'_{01}(s_0)>0$, $p'_{11}(s_0)>0$, $p'_{-10}(s_0)\geq 0$, $p'_{10}(s_0)\geq 0$. One can put $p'_{-10}=p'_{-10}(s_0)$, $p'_{10}=p'_{10}(s_0)$ in any point of the interval and get a system of three equations with three unknown variables $p'_{-11}$, $p'_{01}$, $p'_{11}$, which has a strongly positive solution in $s_0$. Since its coefficients depend continuously on $s$, so does the solution. Thus for sufficiently small $\varepsilon>0$ there exists a solution $p'_{-11}(s_0+\varepsilon)>0$, $p'_{01}(s_0+\varepsilon)>0$, $p'_{11}(s_0+\varepsilon)>0$. This contradicts the fact that $s_0$ is the "last" point.

Thus, the problem is reduced to the case, when at most two probabilities are different from zero. Its verification is purely computational and so we omit it.

Note that $q(1, p_{0-1}/p_{01})<0$, as $\mathsf{E}_y>0$. So the number of zeros of the function $q(s)$ for all parameters in the set $\{p'_{ij}\geq 0 : \sum_{i,j} p'_{ij}=1, q(s_1)>0\}$ [resp. $\{p'_{ij}\geq 0 : \sum_{i,j} p''_{ij}=1, q(s_3)>0\}$] should be the same. Otherwise $q(s)$ would have had a zero of higher order than the zeros for some $p'_{ij}$. This is impossible because of the statement just proved. Similarly, the number of zeros in $\{p'_{ij}\geq 0 : \sum_{i,j} p'_{ij}=1, q(s_1)<0\}$ [resp. $\{p'_{ij}\geq 0 : \sum_{i,j} p''_{ij}=1, q(s_3)>0\}$] is constant. Therefore, checking some special case (e.g. $p'_{11}=p'_{01}=p'_{-11}=0$), one proves the lemma. □

**Proposition 3.2.** *There exist constants $C>0$ and $h=h(i_0,j_0)>0$, such that*

$$\sum_{j=0}^{\infty} \pi_{ij}^{i_0,j_0} \leq C \qquad \text{for all } i \geq 0; \tag{3.29}$$

$$\sum_{j=0}^{\infty} \pi_{ij}^{i_0 j_0} \quad \le \quad \exp(hi) \qquad \textit{for all } i < 0. \tag{3.30}$$

*Proof.* The first inequality is a simple corollary of state homogeneity.

Let us turn to the second. Let $(X_n, Y_n)$ be the position of the chain at time $n$, $X_0 = i_0$, $Y_0 = j_0$. It suffices to show that for some $h = h(i_0, j_0) > 0$

$$\mathsf{P}\Big\{ \bigcup_{n=0}^{\infty} (X_n = i) \Big\} \le \exp(hi), \quad \text{for all } i < 0. \tag{3.31}$$

Since $\mathsf{E}_x$, $\mathsf{E}_y > 0$, one can find $k_0 > 0$ such that $\mathsf{E}(X_{n+k_0} \mid Y_n = 0) \ge \varepsilon > 0$. Let us construct the sequence of stopping times $N_0 := 0$,

$$N_k = \begin{cases} N_{k-1} + 1, & \text{if } Y_{N_{k-1}} \ne 0; \\ N_{k-1} + k_0, & \text{if } Y_{N_{k-1}} = 0. \end{cases}$$

The sequence $X_{N_k}$ satisfies the conditions of Theorem 2.1.8 in [3] with reverse inequality. Then for some $\delta_1, \delta_2 > 0$

$$\mathsf{P}\{X_n = i\} \le \mathsf{P}\{X_n < \delta_1 n\} \le \exp(-\delta_2 n), \tag{3.32}$$

which entails

$$\begin{aligned} \mathsf{P}\Big\{ \bigcup_{n=0}^{\infty} (X_n = i) \Big\} \quad &= \quad \mathsf{P}\Big\{ \bigcup_{n=-i+i_0}^{\infty} (X_n = i) \Big\} \\ &\le \quad \sum_{n=-i+i_0}^{\infty} \mathsf{P}\{X_n = i\} \le \exp(hi). \end{aligned}$$

$\square$

**Lemma 3.7.** *If* $\exp(-h) < |x| < 1$, $|y| < 1$, $|z| \le 1$, *the following equation holds:*

$$Q(x,y,z) \sum_{\substack{i=-\infty \\ j=1}}^{\infty} \pi_{ij}^{i_0 j_0}(z) x^{i-1} y^{j-1} = q(x,y,z) \pi^{i_0 j_0}(x,z) + x^{i_0} y^{j_0}, \tag{3.33}$$

*where*

$$\begin{aligned} Q(x,y,z) \quad &= \quad xy\big(1 - z(p_{10}x + p_{01}y + p_{-10}x^{-1} + p_{0-1}y^{-1})\big), \\ q(x,y,z) \quad &= \quad x\Big(z \sum_{i,j} p'_{ij} x^i y^j - 1\Big), \\ \pi^{i_0 j_0}(x,z) \quad &= \quad \sum_{i=-\infty}^{\infty} \pi_{i0}^{i_0 j_0}(z) x^{i-1}. \end{aligned}$$

*Proof.* We have

$$P_{ij}^{i_0j_0}(t+1) \quad = \quad \sum_{k=-\infty}^{\infty}\sum_{l=1}^{\infty} p_{i-k,j-l}P_{kl}^{i_0j_0}(t) + \sum_{k=-\infty}^{\infty} p'_{i-k,j}P_{k0}^{i_0j_0}(t). \qquad (3.34)$$

Equation (3.34) together with the definition of $\pi_{ij}^{i_0j_0}(z)$ yields

$$\pi_{ij}^{i_0j_0}(z) - \pi_{ij}^{i_0j_0}(0) = z\Big(\sum_{k=-\infty}^{\infty}\sum_{l=1}^{\infty} p_{i-k,j-l}\pi_{kl}^{i_0j_0}(z) + \sum_{k=-\infty}^{\infty} p'_{i-k,j}\pi_{kl}^{i_0j_0}(z)\Big)$$

for $j \geq 1$ and

$$\pi_{i0}^{i_0j_0}(z) - \pi_{i0}^{i_0j_0}(0) = z\Big(\sum_{k=-\infty}^{\infty}\sum_{l=1}^{\infty} p_{i-k,-l}\pi_{kl}^{i_0j_0}(z) + \sum_{k=-\infty}^{\infty} p'_{i-k,0}\pi_{kl}^{i_0j_0}(z)\Big),$$

where $|z| \leq 1$. Let us multiply these equations by $x^iy^j$, where $|y| < 1$, $\exp(-h) < |x| < 1$. Taking the summation over $i, j$ and changing the order of the summation, we get:

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\pi_{ij}^{i_0j_0}(z)x^iy^j + \sum_{i=-\infty}^{\infty}\pi_{i0}^{i_0j_0}(z)x^i - x^{i_0}y^{j_0} \\
&\qquad = \quad z\sum_{i,j} p_{ij}x^iy^j \sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\pi_{ij}^{i_0j_0}(z)x^iy^j + z\sum_{i,j} p'_{ij}x^iy^j \sum_{i=-\infty}^{\infty}\pi_{i0}^{i_0j_0}(z)x^i.
\end{aligned}$$

The sums in the last equation are finite. In fact, due to Proposition 3.2

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\pi_{ij}^{i_0j_0}(1)|y|^j|x|^i &\leq \sum_{i=0}^{\infty} C|x|^i < \infty; \\
\sum_{i=-\infty}^{0}\sum_{j=1}^{\infty}\pi_{ij}^{i_0j_0}(1)|y|^j|x|^i &\leq \sum_{i=-\infty}^{0} \exp(hi)|x|^i < \infty.
\end{aligned}$$

Thus we obtain (3.33). □

**Corollary 3.2.** *For $|x| < 1$, $|y| < 1$ the following equation holds:*

$$Q(x,y)\sum_{\substack{i=-\infty \\ j=1}}^{\infty}\pi_{ij}^{i_0j_0}x^{i-1}y^{j-1} = q(x,y)\pi^{i_0,j_0}(x) + x^{i_0}y^{j_0}, \qquad (3.35)$$

*where*

$$\pi^{i_0j_0}(x) = \sum_{i=-\infty}^{\infty}\pi_{i0}^{i_0j_0}x^{i-1}.$$

*Proof.* This is equation (3.33) with $z = 1$. In accordance with notation (2.1), (2.15)

$$Q(x, y, 1) = Q(x, y), \qquad q(x, y, 1) = q(x, y).$$

□

Let us project (3.35) on the Riemann surface $\mathbf{S}$. Since $Q(x(s), y(s)) = 0$, we have

$$q(x(s), y(s))\pi(x(s)) + x^{i_0}(s)y^{j_0}(s) = 0 \tag{3.36}$$

in the domain $\Delta = \{s : e^{-h} < |x(s)| < 1, |y(s)| < 1\}$.

We put

$$\pi^{i_0 j_0}(s) := \pi^{i_0 j_0}(x(s))$$

in the points $s \in \mathbf{S}$, where $\exp(-h) < |x(s)| < 1$. Our next step is to extend the definition of the function $\pi^{i_0 j_0}(s)$ to the whole $\mathbf{S}$.

*Definition of $\pi^{i_0 j_0}(s)$ on $\mathbf{S}$.*

The Riemann surface is divided by the curves $\{s : x_1 \leq x(s) \leq x_2\}$ and $\{s : x_3 \leq x(s) \leq x_4\}$ into two domains $D_1$ and $D_2$, such that $\Delta \subset D_1$. (In particular the interval $(s_3, s_1) \subset F_0$ belongs to $D_1$ and $(s_1, s_3) \subset F_0$ to $D_2$.) For all $s \in D_1$ there exists a unique $s' \in D_2$, such that $x(s') = x(s)$ and if that, then $y(s') = p_{0-1}/(p_{01}y(s))$. This amounts to saying that $D_1 = \xi D_2$, where $\xi$ is the Galois automorphism (3.6). Let us put

$$\begin{aligned} \pi^{i_0 j_0}(s) &:= -\frac{x^{i_0}(s)y^{j_0}(s)}{q(x(s), y(s))} && \text{for } s \in \overline{D}_1, \\ \pi(s) &:= \pi(\xi s) && \text{for } s \in D_2. \end{aligned} \tag{3.37}$$

This means that

$$\pi^{i_0 j_0}(s) = -\frac{x^{i_0}(s)\left(p_{0-1}/(p_{01}y(s))\right)^{j_0}}{q\left(x(s),\ p_{0-1}/(p_{01}y(s))\right)} \qquad \text{for } s \in D_2. \tag{3.38}$$

The function $\pi^{i_0 j_0}(s)$ is *meromorphic* in $D_1$ and $D_2$. Equation (3.36) holds in $\overline{D}_1$ but in general not in $D_2$.

*Meromorphic continuation of $\pi^{i_0 j_0}(x)$ on $\mathbf{C}$.*

The function $\pi^{i_0 j_0}(x) = \sum_{i=-\infty}^{\infty} \pi_{i0}^{i_0 j_0}(x)$ is holomorphic in $\{x : \exp(-h) < |x| < 1\}$. Setting

$$\pi^{i_0 j_0}(x) := \pi^{i_0 j_0}(s),$$

where $s \in \mathbf{S}$ is such that $x(s) = x$, provides its meromorphic continuation on the whole complex plane cut along the segments $[x_1, x_2]$, $[x_3, x_4]$.

*Remark* 3.3. The function $\pi^{i_0 j_0}(s)$ has no pole at $s_E$, since

$$\pi^{i_0 j_0}(s_E) = -\frac{(p_{0-1}/p_{01})^{j_0}}{q(1, p_{0-1}/p_{01})} < \infty.$$

Consequently, the function $\pi^{i_0 j_0}(x)$ is holomorphic in the domain $\exp(-h) < |x| < 1+\varepsilon$ for sufficiently small $\varepsilon > 0$. In other words, $\sum_{i=-\infty}^{\infty} \pi_{i0}^{i_0 j_0} < \infty$. (This last fact can be also deduced by purely probabilistic techniques, namely martingales.)

**Lemma 3.8.** *For all $j > j_0$ and all $i \in \mathbf{Z}$*

$$\pi_{ij}^{i_0 j_0} = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{q(s)\pi^{i_0 j_0}(s)}{x^i(s) y^j(s)}\, d\omega + \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{x^{i_0}(s) y^{j_0}(s)}{x^i(s) y^j(s)}\, d\omega. \tag{3.39}$$

*If $j \le j_0$, then* (3.39) *holds with the contour $\widetilde{\Gamma}_1$ in the second integral whenever $i > i_0$, and $\widetilde{\Gamma}_0$ whenever $i < i_0$.*

*Proof.* Let us find $\pi_{ij}^{i_0 j_0}$ from equation (3.35) as the coefficients of a Laurent series:

$$\pi_{ij}^{i_0 j_0} = \left(\frac{1}{2\pi i}\right)^2 \int\limits_{|x|=1-\varepsilon} \int\limits_{|y|=1-\varepsilon} \frac{q(x,y)\pi^{i_0 j_0}(x) + x^{i_0} y^{j_0}}{x^i y^j Q(x,y)}\, dy\, dx. \tag{3.40}$$

Given $x$ with $|x| = 1-\varepsilon$, the inner integral

$$\frac{1}{2\pi i} \int\limits_{|y|=1-\varepsilon} \frac{q(x,y)\pi^{i_0 j_0}(x) + x^{i_0} y^{j_0}}{x^i y^j Q(x,y)}\, dy \tag{3.41}$$

equals the sum of the residues at the poles of the integrand

$$\frac{q(x,y)\pi^{i_0 j_0}(x) + x^{i_0} y^{j_0}}{x^i y^j Q(x,y)} \tag{3.42}$$

outside the circle $|y| = 1-\varepsilon$ with "$-$" sign. Whenever $x$ is fixed, the function $Q(x,y)$ has two zeros $Y_0(x)$ and $Y_1(x)$, $Y_0(1-\varepsilon) < Y_1(1-\varepsilon)$. Let us show that

$$|Y_0(x)| < 1-\varepsilon \quad \text{and} \quad |Y_1(x)| > 1-\varepsilon \quad \text{for all } x : |x| = 1-\varepsilon. \tag{3.43}$$

For $|x| = 1$, $x \ne 1$, these inequalities are stated in Lemma 3.1. Thus, it suffices to prove that on the complex plane the smooth closed curve $\{Y_0(x) : |x| = 1-\varepsilon\}$ is inside $h_y(\Gamma_0) = \{Y_0(x) : |x| = 1\}$ and that $\{Y_1(x) : |x| = 1-\varepsilon\}$ is outside $h_y(\Gamma_1)$. Indeed, these curves do not intersect. Otherwise for some pair $x, \widetilde{x}$, $|x| = 1$, $|\widetilde{x}| = 1-\varepsilon$, we would have $Y(x) = Y(\widetilde{x})$, and so $x\widetilde{x} = p_{-10}/p_{10}$. This is impossible for $\varepsilon$ sufficiently small. It is also easily checked explicitly that $Y_0(1-\varepsilon) < Y_0(1) = p_{0-1}/p_{01}$ and $Y_1(1-\varepsilon) > Y_1(1) = 1$. So, continuity of $Y_0(x)$ and $Y_1(x)$ on $x$ gives (3.43).

The poles of the function (3.42) as a function of $y$ can only occur for $x = Y_0(x),\ Y_1(x),\ 0,\ \infty$; $|Y_0(x)| < 1-\varepsilon$, $|Y_1(x)| > 1-\varepsilon$. If $j > j_0$, the residue at infinity is always zero. Then the integral (3.41) equals the residue of the function (3.42) at $Y_1(x)$ with "$-$" sign. Therefore

$$\pi_{ij}^{i_0 j_0} = -\frac{1}{2\pi i} \int\limits_{|x|=1-\varepsilon} \frac{q(x, Y_1(x))\pi^{i_0 j_0}(x) + x^{i_0} Y_1^{j_0}(x)}{x^i Y_1^j(x)[2a(x)Y_1(x) + b(x)]}\, dx. \tag{3.44}$$

In view of Remark 3.3 the integrand in (3.44) is holomorphic in $1-\varepsilon < |x| < 1+\varepsilon$ and thus we can shift the contour to $|x| = 1$. To complete the proof, we take into account the definition of the form $d\omega$ (3.2) and of the curve $\Gamma_1$ (3.1).

When $j \le j_0$, we split (3.40) into two terms and exchange the roles of $x$ and $y$ in the second term. The proof of the lemma is terminated. □

**Lemma 3.9.** *Let* $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$, $\gamma(r) \to \gamma$ *as* $r \to \infty$, *where* $\gamma \in (0, \pi)$ *and let*

$$q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(s(\gamma)) + x^{i_0}(\gamma)y^{j_0}(\gamma) \neq 0.$$

• *If* $0 < \gamma \le \gamma_E$, *i.e.* $s(\gamma) \in (s_1, s_E]$, *assume that the function* $\pi^{i_0 j_0}(s)$ *has no poles on the segment* $[s(\gamma), s_E]$. *Then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{C(\gamma, i_0, j_0)}{\sqrt{j}\, x^i(\gamma(r)) y^j(\gamma(r))} \qquad \text{as } r \to \infty. \tag{3.45}$$

*Here,*

$$\begin{aligned} C(\gamma, i_0, j_0) &= \left[q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(x(\gamma)) + x^{i_0}(\gamma)y^{j_0}(\gamma)\right] \\ &\times \left|x^{\operatorname{ctg}\gamma}(\gamma)y(\gamma)\right|^{1/2} \left[2a(x(\gamma))y(\gamma) + b(x(\gamma))\right]^{-1} \left|\frac{d^2\, x^{\operatorname{ctg}\gamma}(\gamma)Y(x(\gamma))}{dx^2}\right|^{-1/2}. \end{aligned} \tag{3.46}$$

• *If* $\pi > \gamma \ge \gamma_E$, *i.e.* $s(\gamma) \in [s_E, s_3)$, *assume that the function* $\pi^{i_0 j_0}(s)$ *has no poles on the segment* $[s_E, s(\gamma)]$. *Then the asymptotics of* $\pi_{ij}^{i_0 j_0}$ *is given by* (3.45) *with the constant* (3.46).

*Proof.* We start by analysing of the asymptotics of the integral

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{q(x(s), y(s))\pi^{i_0 j_0}(s) + x^{i_0}(s)y^{j_0}(s)}{\left(x^{\operatorname{ctg}\gamma}(s)y(s)\right)^j}\, d\omega, \tag{3.47}$$

as $j \to \infty$ and then use the previous lemma.

To apply the saddle-point method, let us shift the contour $\Gamma_1$ to the curve $\Gamma_\gamma$ as in the proof of Lemma 3.6. (In a sufficiently small neighbourhood of the saddle-point this is the curve of steepest descent for $\ln(x(s)y^{\operatorname{ctg}\gamma}(s))$. It is homological to $\Gamma_1$ and belongs to $D_\gamma^+ = \{s : |x^{\operatorname{ctg}\gamma}(s)y(s)| > x(\gamma)^{\operatorname{ctg}\gamma}y(\gamma)\}$.) Let $E_\gamma$ be a domain on the Riemann surface bounded by $\Gamma_1, \Gamma_\gamma$ and containing the interval $(s(\gamma), s_E) \subset F_0$ if $s(\gamma) \in (s_1, s_E)$ and the interval $(s_E, s(\gamma))$ if $s(\gamma) \in (s_E, s_3)$, as in Lemma 3.6. Denote by $s_1, s_2, \ldots, s_n$ the poles of the integrand in $E_\gamma$, if they exist. They can only occur at the poles of $\pi^{i_0 j_0}(s)$. Then, due to Cauchy's theorem

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{q(x(s), y(s))\pi^{i_0 j_0}(s) + x^{i_0}(s)y^{j_0}(s)}{\left(x^{\operatorname{ctg}\gamma}(s)y(s)\right)^j}\, d\omega$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma_\gamma} \frac{q(x(s),y(s))\pi^{i_0j_0}(s) + x^{i_0}(s)y^{j_0}(s)}{\left(x^{\operatorname{ctg}\gamma}(s)y(s)\right)^j}\,d\omega \qquad (3.48)$$

$$+\sum_{k=1}^{n} \frac{q(x(s_k),y(s_k))\operatorname{res}_{x(s_k)}\pi^{i_0j_0}(x)}{\left(x(s_k)^{\operatorname{ctg}\gamma}y(s_k)\right)^j[2(a(x(s_k))y(s_k)+b(x(s_k))]}.$$

The saddle-point method allows to find the asymptotics of the integral along $\Gamma_\gamma$ in (3.48)

$$\frac{1}{\left(x^{\operatorname{ctg}\gamma}(\gamma)y(\gamma)\right)^j}\Big(\sum_{k=0}^{n} c_k(\gamma)j^{-k-1/2} + o(j^{-k-1/2})\Big) \qquad (3.49)$$

as $j\to\infty$, uniformly in a neighbourhood of $\gamma$ (see Theorem 1.7 in [4, Chapter IV]).

Let us turn now to the sum over the poles in (3.48). We call the level curve $\{s : |x^{\operatorname{ctg}\gamma}(s)y(s)| = c_1\}$ "higher" [resp. "lower"] than $\{s : |x^{\operatorname{ctg}\gamma}(s)y(s)| = c_2\}$ if $c_1 > c_2$ [resp. $c_1 < c_2$]. We will also call the point $s$ "higher" [resp. "lower"] than $\widetilde{s}$ if $|x^{\operatorname{ctg}\gamma}(s)y(s)| > |x^{\operatorname{ctg}\gamma}(\widetilde{s})y(s)|$ [resp. "$<$"]. Hence, by (3.49) all poles of $\pi^{i_0j_0}(s)$ among $s_1, s_2, \ldots, s_n$ "higher" than the saddle-point do not contribute to the asymptotics of (3.48), as $j\to\infty$. Let us prove the following proposition.

**Proposition 3.3.** *Assume that there are poles of the function $\pi^{i_0j_0}(s)$ in $E_\gamma$ "lower" than the saddle-point or at the same level. Then the "lowest" of them is on $F_0$ and there are no other poles at the same level.*

*Proof.* Let first $s(\gamma)\in(s_1, s_E)$. We reduce the statement to the corresponding one on the complex plane $\mathbf{C}_x$. If $\exp(-h) < |x| < 1+\varepsilon$, then

$$h_x\pi^{i_0j_0}(s) = \sum_{i=-\infty}^{0} \pi_{i0}^{i_0j_0}x^i + \sum_{i=0}^{\infty}\pi_{i0}^{i_0j_0}x^i,$$

where the first sum is holomorphic in $|x| > \exp(-h)$ and the second one in $|x| < 1+\varepsilon$. The domain $h_xE_\gamma$ being outside the circle $|x| = 1$, the poles of $h_x\pi(s)$ are at the poles of the second sum.

It follows from structural stability, that all level curves

$$\Gamma(s^*,\gamma) = \{s : |x^{\operatorname{ctg}\gamma}(s)y(s)| = x^{\operatorname{ctg}\gamma}(s^*)y(s^*),\ s^*\in F_0\},$$

"lower" than the saddle-point and passing through $E_\gamma$ are homological to $\Gamma_1$ and intersect with $F_0$ at exactly one point. Moreover, if $s(\gamma)\le s^* < s^{**}\le s_E$, i.e. $x(s^*) > x(s^{**})$, then $s^*$ is "higher" than $s^{**}$. We will show that the images $h_x\Gamma(\gamma, s^*)$ of the level curves in question lie inside the circle $|x| = x(s^*)$, except for the point $x(s^*)$ itself. (In other words, for all $s\in\Gamma(s^*,\gamma)$, $s\ne s^*$, $|x(s)| < x(s^*)$. If $s^* = s(\gamma)$, we prove this property only for the component of the level curve which is in $E_\gamma$.) Then the result follows immediately from structural stability and the Hadamard–Pringsheim theorem. This theorem states that the first singularity of the function $\sum_{i=0}^{\infty} a_ix^i$, $a_i\ge 0$, occurs at a real point

$r > 0$ and it is a pole. (Hence, the minimal $x^* > 1 + \varepsilon$ on the real axis, where there is a pole of $\sum_{i=0}^{\infty} \pi_{ij}^{i_0 j_0} x^i$ is exactly the projection on $\mathbf{C}_x$ of the "lowest" pole of $\pi^{i_0 j_0}(s)$ in $E_\gamma$.)

Let us show that $h_x(\Gamma(\gamma, s^*))$ and the circle $|x| = x(s^*)$ intersect only at $x = x(s^*)$. Suppose that there exists another point $s \in \Gamma(\gamma, s^*)$, not on $F_0$, such that $|x(s)| = x(s^*)$. Then $|y(s)| = y(s^*)$ and $\sum_{i,j} p_{ij} x(s)^i y(s)^j = \sum_{i,j} p_{ij} x(s^*)^i y(s^*)^j = 1$. By simple considerations of sums of complex numbers, this can only occur if $x(s) = x(s^*)$, $y(s) = y(s^*)$.

The set $h_x(\Gamma(\gamma, s^*))$ being a smooth closed curve, it suffices now to find one point $s \in \Gamma(\gamma, s^*)$, such that $|x(s)| < x(s^*)$. Let us take the point $\widetilde{s}(\gamma)$, where $\Gamma(\gamma, s^*)$ intersects with $F_1$. Then $\widetilde{x}(\gamma) := x(\widetilde{s}(\gamma)) < 0$, $\widetilde{y}(\gamma) := y(\widetilde{s}(\gamma)) > 0$, and $-\widetilde{x}(\gamma)\widetilde{y}(\gamma)^{\operatorname{tg}\gamma} = x(s^*)y(s^*)^{\operatorname{tg}\gamma}$. Consequently,

$$\frac{d\widetilde{x}}{d\gamma}(0) = -\widetilde{x}(0) \ln \widetilde{y}(0) - x(s^*) \ln y(s^*) > 0.$$

In fact, it is easy to see that $\widetilde{y}(0) > 1$ and $\widetilde{x}(0) < 0$, $y(s^*) > 1$, $x^* > 0$. Then $\widetilde{x}(\gamma)$ is inside the circle for sufficiently small $\gamma > 0$. Since $\widetilde{x}(\gamma)$ depends on $\gamma$ continuously and never coincides with $-x(s^*)$, we may extend the proposition to all $\gamma$.

Let now assume $s(\gamma) \in (s_E, s_3)$. The image $h_x E_\gamma$ of the domain $E_\gamma$ being inside the circle $|x| = 1$ in this case, the poles of $h_x \pi^{i_0 j_0}(s)$ are the poles of $\sum_{i=-\infty}^{0} \pi_{ij}^{i_0 j_0} x^{-is}$. By structural stability the point $s^*$ is "lower" than $s^{**}$ if $s_E \le s^* < s^{**} \le s(\gamma)$, i.e. if $x(s^*) < x(s^{**})$. Proceeding along the same lines as in the case above, one can deduce that the images of the level curves $h_x \Gamma_\gamma(\gamma, s^*)$, when $s^* \in [s_E, s(\gamma)]$, are outside the circles $|x| = x(s^*)$. Then the result follows again from structural stability and the Hadamard−Pringsheim theorem. □

Let us continue the proof of Lemma 3.9. By assumption, there are no poles of $\pi^{i_0 j_0}(s)$ on the segment $[s(\gamma), s_E]$ [resp. $[s_E, s(\gamma)]$]. Then due to this proposition there are no poles in $E_\gamma$ "lower" than the saddle-point or at its level. Then the asymptotics of (3.48) is (3.49). Moreover $c_0(\gamma) = C(\gamma, i_0, j_0)$. Taking into account Lemma 3.8 and the uniformity in (3.49) we obtain (3.45). The proof of the lemma is concluded. □

*Remark* 3.4. It is worthwhile to note that if $\gamma = \gamma_E$ the asymptotics of $\pi_{ij}^{i_0 j_0}$ is $Cj^{-1/2}$.

**Proposition 3.4.** *Assume that the function $\pi^{i_0 j_0}(s)$ has no poles on $[s(\gamma), s_E]$, if $0 < \gamma < \gamma_E$ and that it has no poles on $[s_E, s(\gamma)]$, if $\gamma_E < \gamma < \pi$. Then for all pairs $(i_0, j_0)$*

$$q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(s(\gamma)) + x^{i_0}(\gamma) y^{j_0}(\gamma) \neq 0. \tag{3.50}$$

*Proof.* For a given $\gamma$, let us construct a pair $(i_0, j_0)$ such that (3.50) holds. The point $s(\gamma)$ belongs to the domain $D_2$ for $\gamma \in (0, \pi)$, which domain has been introduced to continue $\pi^{i_0 j_0}(s)$ to all of $\mathbf{S}$. In view of the definition (3.38) of $\pi^{i_0 j_0}(s)$:

$$\begin{aligned}\pi^{i_0 j_0}(s(\gamma)) = \pi^{i_0 j_0}(\xi s(\gamma)) &= -\frac{x^{i_0}(s(\gamma)) y^{j_0}(\xi s(\gamma))}{q(\xi s(\gamma)))} \\ &= -\frac{x^{i_0}(\gamma)\big(p_{0-1}/(p_{01} y(\gamma))\big)^{j_0}}{q\big(x(\gamma),\ p_{0-1}/(p_{01} y(\gamma))\big)}. \end{aligned} \tag{3.51}$$

Then

$$\begin{aligned} & q(x(\gamma), y(\gamma)) \pi^{i_0 j_0}(s(\gamma)) + x^{i_0}(\gamma) y^{j_0}(\gamma) \\ & \quad = \ x^{i_0}(\gamma) y^{j_0}(\gamma) \Big(1 - \frac{q(s(\gamma))}{q(\xi s(\gamma))} \Big(\frac{p_{0-1}}{p_{01} y^2(\gamma)}\Big)^{j_0}\Big). \end{aligned}$$

For all $\gamma \in (0, \pi)$, $y(\gamma) < \sqrt{p_{0-1}/p_{01}}$. Then for $j_0$ sufficiently large we get (3.50).

To derive (3.50) for all pairs $(i_0, j_0)$, the reasoning is completely the same as in Proposition 3.1 and we skip it. □

By virtue of Proposition 3.4 condition (3.50) has become superfluous for the result of Lemma 3.9 to hold. This Lemma deals with the case $\gamma \in (0, \pi)$. However, if $\gamma = 0$ or $\pi$, inequality (3.50) does not hold for any pair $(i_0, j_0)$. The following proposition is devoted to these two particular cases.

**Proposition 3.5.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$, $\gamma(r) \to \gamma$ as $r \to \infty$.*

• *Assume that $\gamma = 0$. If $q(s_1) \neq 0$ and the function $\pi^{i_0 j_0}(s)$ has no poles on the interval $(s_1, s_E)$, then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{1}{x^i(\gamma(r)) y^j(\gamma(r))} \Big( \frac{\widetilde{C}(\gamma(r), i_0, j_0)}{\sqrt{|i|}} + \frac{C_2(\gamma(r), i_0, j_0)}{|i|\sqrt{|i|}} \Big), \tag{3.52}$$

*where*

$$\begin{aligned} & \widetilde{C}(\gamma, i_0, j_0) = \big[x^{i_0}(\gamma) y^{j_0}(\gamma) + q(x(\gamma), y(\gamma)) \pi^{i_0 j_0}(x(\gamma))\big] \\ & \quad \times \big| x(\gamma) y^{\operatorname{tg}\gamma}(\gamma)\big|^{1/2} \big[2\widetilde{a}(y(\gamma)) x(\gamma) + \widetilde{b}(y(\gamma))\big]^{-1} \Big| \frac{d^2\, X(y(\gamma)) y^{\operatorname{tg}\gamma}(\gamma)}{dy^2} \Big|^{-1/2}. \end{aligned}$$

*Moreover,*

$$\begin{aligned} & \lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = \lim_{\gamma \to 0} \frac{C(\gamma, i_0, j_0)}{C(\gamma, 0, 0)} \\ & \quad = \frac{x_3^{i_0} (\sqrt{p_{0-1}/p_{01}})^{j_0} \left( j_0 \sqrt{p_{0-1}/p_{01}}\, q(x_3, \sqrt{p_{0-1}/p_{01}}) - p'_{11} x_3^2 - p'_{01} x_3 - p'_{0-1}\right)}{p'_{11} x_3^2 + p'_{01} x_3 + p'_{0-1}}, \end{aligned} \tag{3.53}$$

*where $C(\gamma, i_0, j_0)$ is defined by Lemma 3.9.*

• *Assume that $\gamma = \pi$. If $q(s_3) \neq 0$ and the function $\pi^{i_0 j_0}(s)$ has no poles on the interval $(s_E, s_3)$, then the asymptotics of $\pi_{ij}^{i_0 j_0}$ is given by (3.52) and (3.53) holds, where $x_3$ is replaced by $x_2$.*

*Proof.* The crucial idea is again to shift the contour to $\Gamma_\gamma$ in the integral (3.47) and apply the saddle-point method. Let us outline some details.

First, the integral (3.47) should be split into two terms, where the contour in the second term is $\widetilde{\Gamma}_1$ or $\widetilde{\Gamma}_0$ according to Lemma 3.8. After shifting the contour in each term we sum the results and get one integral along $\Gamma_\gamma$. Second, the saddle-points $s(0) = s_1$, $s(\pi) = s_3$ are branch points for $x(s)$. Then we should consider $h_y(\Gamma_\gamma)$ on the complex plane $\mathbf{C}_y$. Third, the integrand equals zero at the saddle-point, thus $\widetilde{C}(0, i_0, j_0) = 0$, $\widetilde{C}(\pi, i_0, j_0) = 0$ and we should take into account the second term of the asymptotics as in (3.52). The other details are similar to Lemma 3.9.

Finally, using the L'Hôpital's rule we have

$$\lim_{\gamma\to 0} \frac{\widetilde{C}(\gamma, i_0, j_0)}{\widetilde{C}(\gamma, 0, 0)} = \frac{C_2(0, i_0, j_0)}{C_2(0, 0, 0)}.$$

Then for $\gamma = 0$

$$\begin{aligned}\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) &= \lim_{r\to\infty} \frac{\widetilde{C}(\gamma(r), i_0, j_0)i + C_2(\gamma(r), i_0, j_0)}{\widetilde{C}(\gamma(r), 0, 0)i + C_2(\gamma(r), 0, 0)} \\ &= \lim_{\gamma\to 0} \frac{\widetilde{C}(\gamma, i_0, j_0)}{\widetilde{C}(\gamma, 0, 0)} = \lim_{\gamma\to 0} \frac{C(\gamma, i_0, j_0)}{C(\gamma, 0, 0)}.\end{aligned}$$

The same is true for $\gamma = \pi$. □

Let us now study the case, when $\pi^{i_0 j_0}(s)$ has poles on $(s_1, s_3)$.

**Lemma 3.10.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and $\gamma(r) \to \gamma$ as $r \to \infty$; where $\gamma \in [0, \pi]$.*

- *If $0 \le \gamma < \gamma_E$, i.e. $s(\gamma) \in [s_1, s_E)$, and the function $\pi^{i_0 j_0}(s)$ has exactly one pole $s'$ on the interval $(s(\gamma), s_E) \subset F_0$, $q(s') \neq 0$ and $\mathrm{res}_{x(s')}\, \pi^{i_0 j_0}(x) \neq 0$, then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{q(x(s'), y(s'))\, \mathrm{res}_{x(s')}\, \pi(x)}{x^i(s') y^j(s')[2a(x(s'))y(s') + b(x(s'))]}. \tag{3.54}$$

- *If $\gamma_E < \gamma \le \pi$, i.e. $s(\gamma) \in (s_E, s_3]$, and the function $\pi^{i_0 j_0}(s)$ has exactly one pole $s''$ on the interval $(s_E, s(\gamma)) \subset F_0$, $q(s'') \neq 0$, $\mathrm{res}_{x(s'')}\, \pi^{i_0 j_0}(x) \neq 0$, then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{q(x(s''), y(s''))\, \mathrm{res}_{x(s'')}\, \pi(x)}{x^i(s'') y^j(s'')[2a(x(s''))y(s'') + b(x(s''))]}. \tag{3.55}$$

*Proof.* Proceeding exactly as in Lemma 3.9 we obtain (3.48). By structural stability the pole $s'$ [resp. $s''$] is "lower" than the saddle-point. Since there are no other poles on $[s(\gamma), s_E]$ [resp. $[s_E, s(\gamma)]$], Proposition 3.3 ensures that $s'$ [resp. $s''$] is the "lowest" pole in $E_\gamma$. Then the asymptotics of (3.48) is determined by this and uniform in a neighbourhood of $\gamma$. Using Lemma 3.8, we have the result. □

**Lemma 3.11.** *The function $\pi^{i_0j_0}(s)$ has a pole in $s' \in (s_1, s_E)$ if and only if $q(\xi s') = 0$. This holds if and only if $q(s_1) > 0$. This pole on the interval $(s_1, s_E)$ is unique.*

*The function $\pi^{i_0j_0}(s)$ has a pole in $s'' \in (s_E, s_3)$ if and only if $q(\xi s') = 0$. This holds if and only if $q(s_3) > 0$. This pole on the interval $(s_E, s_3)$ is unique.*

*Moreover, $q(s') \neq 0$, $q(s'') \neq 0$, $\operatorname{res}_{x(s')} \pi^{i_0j_0}(x) \neq 0$, $\operatorname{res}_{x(s'')} \pi^{i_0j_0}(x) \neq 0$.*

*Proof.* In accordance with the definition of the function $\pi^{i_0j_0}(s)$ on these intervals:

$$\begin{aligned}\pi^{i_0j_0}(s) = \pi(\xi s) &= -\frac{x^{i_0}(s)y^{j_0}(\xi s)}{q(\xi s)} \\ &= -\frac{x^{i_0}(s)\big(p_{0-1}/(p_{01}y(s))\big)^{j_0}}{q\big(x(s),\ p_{0-1}/(p_{01}y(s))\big)}.\end{aligned}$$

Then $s' \in (s_1, s_E)$ [resp. $s'' \in (s_E, s_3)$] is a pole of $\pi^{i_0j_0}(s)$ if and only if $\xi s \in ((1, p_{0-1}/p_{01})\,,\ s_1)$ [resp. $\xi s'' \in ((p_{-10}/p_{01}, 1)\,,\ s_3)$] is a zero of $q(s)$. Therefore the result is implied by Lemma 2.5. In addition, it is shown in the proof of this lemma that the zeros are of the first order. Hence, $\operatorname{res}_{x(s')} \pi^{i_0j_0}(x) \neq 0$ and $\operatorname{res}_{x(s'')} \pi^{i_0j_0}(x) \neq 0$.

Moreover,

$$q(s) - q(\xi s) = (y(s) - p_{0-1}/(p_{01}y(s))) \sum_i p'_{i1}x^i.$$

Consequently, if $q(\xi s) = 0$, $s \neq s_1, s_3$, $p'_{-11} + p'_{01} + p'_{11} \neq 0$, then $q(s) \neq 0$. □

**Proposition 3.6.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$, $\gamma(r) \to \gamma$ as $r \to \infty$.*

*Assume that the function $\pi^{i_0j_0}(s)$ has a pole in $s(\gamma) \in (s_1, s_E)$ and no poles on $(s(\gamma), s_E)$. Then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = x(\gamma)^{i_0}\left(p_{0-1}/p_{01}y(\gamma)\right)^{j_0}. \tag{3.56}$$

*The same is true if the function $\pi^{i_0j_0}(s)$ has a pole in $s(\gamma) \in (s_E, s_3)$ and no poles on $(s_E, s(\gamma))$.*

*Proof.* Note that $q(x(\gamma), \xi y(\gamma)) = 0$ by the previous lemma. Let us shift the contour in (3.47) to $\gamma(r)$ as in Lemma 3.9. Consider the integrand in the neighbourhood of $s(\gamma)$ and its projection onto the complex plane $\mathbf{C}_x$. It can be split into two terms:

$$\begin{aligned} & h_x\Big(\frac{q(s)\pi^{i_0j_0}(s) + x^{i_0}(s)y^{j_0}(s)}{x^i(s)y^j(s)}\Big) \\ &\quad = \big[-q(x, Y_1(x))x^{i_0}Y_0^{j_0}(x) + q(x, Y_0(x))x^{i_0}Y_1^{j_0}(x)\big]\frac{1}{x^iY_1^j(x)q(x, Y_0(x))} \\ &\quad = \big[-q(x(\gamma), y(\gamma))x(\gamma)^{i_0}(\xi y(\gamma))^{j_0}\big]\frac{\operatorname{res}_{x(\gamma)} q^{-1}(x, Y_0(x))}{x^iY_1^j(x)(x - x(\gamma))} + \frac{f(x, i_0, j_0)}{x^iY_1^j(x)},\end{aligned}$$

where the branches $Y_0(x)$ and $Y_1(x)$ are such that $Y_1(x(\gamma)) = y(\gamma)$, $Y_0(x(\gamma)) = \xi y(\gamma)$ and the function $f(x, i_0, j_0)$ has no pole at $x(\gamma)$. Then the asymptotics of the integral (3.47) is determined by the asymptotics of the integral over the first term. The result comes from the definition of the Martin kernel. □

*Proof of Theorem* 2.2. We rely on the definition of the Martin kernel and all previous lemmas and propositions.

1. If $q(s_1) = q(x_3, \sqrt{p_{0-1}/p_{01}}) < 0$ and $q(s_3) = q(x_2, \sqrt{p_{0-1}/p_{01}}) < 0$, then by Lemma 3.11 the function $\pi^{i_0 j_0}(s)$ has no poles on the segment $(s_1, s_3)$. Hence, inequality (3.50) holds for all $\gamma \in (0, \pi)$ and Lemma 3.9 applies. Thus,

$$\begin{aligned}\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) &= \lim_{r\to\infty} \frac{\pi_{ij}^{i_0 j_0}}{\pi_{ij}^{00}} = \frac{C(\gamma, i_0, j_0)}{C(\gamma, 0, 0)} \\ &= \frac{q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(s(\gamma)) + x^{i_0}(\gamma) y^{j_0}(\gamma)}{q(x(\gamma), y(\gamma))\pi^{00}(s(\gamma)) + 1}.\end{aligned}$$

Next, recall the definition (3.38) of the function $\pi^{i_0 j_0}(s)$ in $s(\gamma) \in (s_1, s_3) \subset D_2$. Then (2.21) is fulfilled. For $\gamma = 0, \pi$ Proposition 3.5 is applicable.

2. If $q(s_1) = q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$ and $q(s_3) = q(x_2, \sqrt{p_{0-1}/p_{01}}) < 0$, then by Lemma 3.11 there is exactly one pole $s'$ on $(s_1, s_E)$, $q(\xi s') = 0$, and no poles on $(s_E, s_3)$. In accordance with notations of Subsection 2.3 we have $x' = x(\xi s')$, $y' = y(\xi s')$ and the angle $\gamma' \in (0, \gamma_E)$ such that $s(\gamma') = s'$, i.e. $x(\gamma') = x'$, $y(\gamma') = p_{0-1}/p_{01} y'$. For $\gamma \in [0, \gamma')$ Lemma 3.10 is applicable. Therefore, by the definition of $\pi^{i_0 j_0}(x)$ on the complex plane we have

$$\begin{aligned}\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) &= \lim_{r\to\infty} \frac{\pi_{ij}^{i_0 j_0}}{\pi_{ij}^{00}} = \frac{\operatorname{res}_{x(s')} \pi^{i_0 j_0}(x)}{\operatorname{res}_{x(s')} \pi^{00}(x)} \\ &= \frac{(x')^{i_0}(y')^{j_0} \operatorname{res}_{x'} q^{-1}(x, Y(x))}{\operatorname{res}_{x'} q^{-1}(x, Y(x))} = (x')^{i_0}(y')^{j_0}.\end{aligned}$$

To find the asymptotics of the Martin kernel when $\gamma = \gamma'$, $\gamma \in (\gamma', \pi)$ or $\gamma = \pi$, we use Proposition 3.6, Lemma 3.9 and Proposition 3.5 respectively.

3. If $q(s_1) = q(x_3, \sqrt{p_{0-1}/p_{01}}) < 0$ and $q(s_3) = q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$, then by Lemma 3.11 there is exactly one pole on $s''$ on $(s_E, s_3)$, $q(\xi s'') = 0$, and no poles on $(s_1, s_E)$. Define the angle $\gamma''$ such that $s(\gamma'') = s''$, as in the above case. For $\gamma \in [0, \gamma'')$ Lemma 3.10 and Proposition 3.5 apply and for $\gamma \in [\gamma''; \pi]$ Lemma 3.9 and Proposition 3.6.

4. If $q(s_1) = q(x_3, \sqrt{p_{0-1}/p_{01}}) > 0$ and $q(s_3) = q(x_2, \sqrt{p_{0-1}/p_{01}}) > 0$, then by Lemma 3.11 there is exactly one pole $s'$ on $(s_1, s_E)$ and exactly one pole $s''$ on $(s_E, s_3)$. Define the angles $\gamma'$, $\gamma''$ as in the previous cases. Then for $\gamma \in (\gamma', \gamma'')$ Lemma 3.9 and for $\gamma \in [0, \gamma')$, $\gamma \in (\gamma'', \pi]$ Lemma 3.10 apply. For $\gamma = \gamma', \gamma''$ we use Proposition 3.6.

To show that the Martin boundary is minimal, the arguments are the same as in the case of the plane. (The necessary remarks on the asymptotics of the harmonic functions obtained, have already been made in the proof of Proposition 3.4.) □

### 3.4. Random walk in $\mathbf{Z}_+ \times \mathbf{Z}$, $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$: proofs

*Proof of Lemma* 2.6. One can rephrase this lemma as follows: *The function* $q(s)$ *has a zero on the interval* $(s_3\,,\,(1,1))$ *if and only if* $q(s_3) > 0$. *This zero is unique.* Moreover, *this zero is of the first order.*

First of all, we show that the zero of the function $q(s)$ on the interval $(s_3, (1,1))$, if it exists at all, should be of the first order for all parameters $\{p'_{ij}\}$. One can do this by proceeding along the same lines as in the proof of Lemma 2.5.

Note that $q(1,1) = 0$, $q'_x(1,1) = \mathsf{E}_y^{-1}(\mathsf{E}_y\,\mathsf{E}'_x - \mathsf{E}'_y\,\mathsf{E}_x) > 0$. Then the number of zeros of $q(s)$ on the interval in question should be the same for all parameters from the set $\{p'_{ij} : \sum_{i,j} p'_{ij} = 1, q(s_3) > 0\}$ and for all parameters from $\{p'_{ij} : \sum_{i,j} p'_{ij} = 1, q(s_3) < 0\}$. (Otherwise, for some $\{p'_{ij}\}$ there is a zero of multiplicity more than one.) Checking some simple case (e.g. when only two parameters $p'_{ij}$ are non-zero), we have the lemma. □

Proposition 3.2 remains valid in this case.

*Proof of Proposition* 3.2. We show again (3.32) to get (3.31). Let $N_0 := 0$, $N_k = \min\{n > N_{k-1} :\ Y_n = 0\}$. The well-known estimates of sums of i.i.d. random variables with exponentional tails yield that $N_k < \infty$ a.s. Moreover, $N_{k+1} - N_k$, $k \geq 1$, are i.i.d. random variables with mean $-\mathsf{E}'_y\,/\,\mathsf{E}_y$ and

$$\mathsf{P}\{N_2 - N_1 > n\} \leq \exp(-\delta_1 n) \quad \text{for some } \delta_1 > 0. \tag{3.57}$$

Then by the general Kolmogorov inequality

$$\mathsf{P}\left\{\bigcup_{l=0}^{k} |N_l + l(\mathsf{E}'_y\,/\,\mathsf{E}_y)| > k\varepsilon\right\} \leq \exp(-\delta_2 k) \quad \text{for some } \delta_2 > 0. \tag{3.58}$$

The sequence $X_{N_k}$ consists again of sums of i.i.d. random variables $X_{N_k} - X_{N_{k-1}}$ with exponentional tails. Then

$$\mathsf{P}\left\{|X_{N_k} - (\mathsf{E}'_x - \mathsf{E}_x(\mathsf{E}'_y\,/\,\mathsf{E}_y))| > k\varepsilon\right\} \leq \exp(-\delta_3 k) \quad \text{for some } \delta_3 > 0. \tag{3.59}$$

For a fixed $n$, let us define $\tau_n = \max\{N_k : N_k \leq n\}$, i.e. $\tau_n = N_k$ if $N_k \leq n < N_{k+1}$. Let $k_0 = [n/(-\mathsf{E}'_y\,/\,\mathsf{E}_y - \varepsilon) - 1]$. Then for $i < 0$

$$\begin{aligned}
\mathsf{P}\{X_n = i\} \quad &\leq \quad \mathsf{P}\{\tau_n = N_k \text{ for } k < k_0\} \\
&\quad + \sum_{k=k_0}^{\infty} \mathsf{P}\{(\tau_n = N_k) \cap (X_{N_k} > ck) \cap (n - \tau_n > ck)\} \\
&\quad + \sum_{k=k_0}^{\infty} \mathsf{P}\{(\tau_n = N_k) \cap (X_{N_k} < ck)\} \\
&\leq \quad \mathsf{P}\{N_{k+1} > n \text{ for some } k < k_0\}
\end{aligned} \tag{3.60}$$

$$+\sum_{k=k_0}^{\infty} \mathsf{P}\{N_{k+1}-N_k > ck\} + \sum_{k=k_0}^{\infty} \mathsf{P}\{X_{N_k} < ck\},$$

where $c = \mathsf{E}_x' - \mathsf{E}_x \, \mathsf{E}_y' \,/\, \mathsf{E}_y - \varepsilon > 0$. We estimate the first term in (3.60) by (3.58) the second one by (3.57) and the third one by (3.59). Whence (3.32) holds. □

Lemma 3.7 is true as well. Its proof is completely the same as in the previous subsection, provided by Proposition 3.2. Then equation (3.35) holds.

On the Riemann surface $Q(x(s), y(s)) = 0$. Thus

$$q(x(s), y(s))\pi(x(s)) + x^{i_0}(s)y^{j_0}(s) = 0 \tag{3.61}$$

in the domain $\Delta = \{s : e^{-h} < |x(s)| < 1, |y(s)| < 1\}$. Let us put

$$\pi^{i_0 j_0}(s) := \pi^{i_0 j_0}(x(s))$$

at the points $s \in \mathbf{S}$, where $\exp(-h) < |x(s)| < 1$.

*Definition of $\pi^{i_0 j_0}(s)$ on $\mathbf{S}$.*

This procedure is the same as in the case $\mathsf{E}_x > 0, \mathsf{E}_y > 0$. Let us divide the Riemann surface by the curves $\{s : x_1 \le x(s) \le x_2\}$ and $\{s : x_3 \le x(s) \le x_4\}$ into two domains $D_1$ and $D_2$, such that $\Delta \subset D_1$. We have again $D_1 = \xi D_2$, where $\xi$ is the Galois automorphism (3.6). Let us put

$$\begin{aligned} \pi^{i_0 j_0}(s) \quad &:= \quad -\frac{x^{i_0}(s)y^{j_0}(s)}{q(x(s), y(s))} \qquad &&\text{for } s \in \overline{D}_1, \\ \pi(s) \quad &:= \quad \pi(\xi s) &&\text{for } s \in D_2. \end{aligned} \tag{3.62}$$

This means that

$$\pi^{i_0 j_0}(s) = -\frac{x^{i_0}(s)\left(p_{0-1}/(p_{01}y(s))\right)^{j_0}}{q\left(x(s),\ p_{0-1}/(p_{01}y(s))\right)} \qquad \text{for } s \in D_2. \tag{3.63}$$

The function $\pi^{i_0 j_0}(s)$ is *meromorphic* in $D_1$ and $D_2$. Equation (3.61) holds in $\overline{D}_1$ but generally does not hold in $D_2$.

*Meromorphic continuation of $\pi^{i_0 j_0}(x)$ on $\mathbf{C}_x$.*

The function $\pi^{i_0 j_0}(x)$ is defined and holomorphic on the domain $\exp(-h) < |x| < 1$. Setting

$$\pi^{i_0 j_0}(x) := \pi^{i_0 j_0}(s), \quad \text{where } s \in \mathbf{S} \text{ is such that } x(s) = x,$$

we meromorphically continue it on the whole complex plane cut along the segments $[x_1, x_2]$, $[x_3, x_4]$.

*Remark* 3.5. It is important to emphasize that the function $\pi^{i_0 j_0}(s)$ has a pole at the point $s_E^* = (1, p_{0-1}/p_{01}) = \Gamma_1 \cap F_0$. In fact, $s_E^* \in D_2$, then $\pi^{i_0 j_0}(s_E^*) = -q^{-1}(1,1) = \infty$, since $q(1,1) = 0$. Consequently, on the complex plane $\mathbf{C}_x$ the function $\pi^{i_0 j_0}(x)$ has a pole in $x = 1$, i.e. $\sum_{i=-\infty}^{\infty} \pi_{i0}^{i_0 j_0} = \infty$. This is a crucial difference from the case $\mathsf{E}_x, \mathsf{E}_y > 0$.

One can get this last fact by purely probabilistic techniques. Moreover, there exists a constant $C$ such that $\pi_{i0}^{i_0 j_0} \to C$, as $i \to +\infty$.

Our next step is to represent $\pi_{ij}^{i_0 j_0}$ as an integral on the Riemann surface along a curve, which we denote by $\Gamma_{1-\varepsilon}$. Let us define it.

The algebraic function $Y(x)$ has two branches $Y_0(x)$ and $Y_1(x)$, $Y_0(1) < Y_1(1)$, on the circle $|x| = 1-\varepsilon$, $\varepsilon > 0$. For all $x$, such that $|x| = 1-\varepsilon$, $|Y_1(x)| > 1$ since this holds for $|x| = 1$. However, if $\mathsf{E}_x < 0, \mathsf{E}_y < 0$, (3.43) is not true: $|Y_0(x)|$ may be both greater and less than 1 on $|x| = 1-\varepsilon$. Nevertheless, there exists $\delta > 0$ such that $|Y_0(x)| \le 1+\delta$, $|Y_1(x)| > 1+\delta$ if $|x| = 1-\varepsilon$ for all sufficiently small $\varepsilon$. (This is a corollary of Lemma 3.1 and simple continuity arguments.) Let us define

$$\Gamma_{1-\varepsilon} = h_x^{-1}\{|x| = 1-\varepsilon\} \cap \{s : |y(s)| > 1+\delta\}.$$

**Lemma 3.12.** *For all $j > j_0$ and all $i \in \mathbf{Z}$*

$$\pi_{ij}^{i_0 j_0} = \frac{1}{2\pi i} \int\limits_{\Gamma_{1-\varepsilon}} \frac{q(s)\pi^{i_0 j_0}(s)}{x^{i_0}(s) y^{j_0}(s)}\, d\omega + \frac{1}{2\pi i} \int\limits_{\Gamma_{1-\varepsilon}} \frac{x^{i_0}(s) y^{j_0}(s)}{x^{i}(s) y^{j}(s)}\, d\omega. \tag{3.64}$$

*If $j \le j_0$, (3.64) holds with $\widetilde{G}_1$ the contour in the second integral whenever $i > i_0$, and $\widetilde{G}_0$ whenever $i < i_0$.*

*Proof.* It is similar to the proof of Lemma 3.8. We will emphasize the details, that are different.

Let $j > j_0$. Equation (3.35) allows to represent $\pi_{ij}^{i_0 j_0}$ as the double integral (3.40). Our goal is to find the inner integral (3.41) as a sum of the residues of the integrand (3.42) at the poles outside the circle $|y| = 1-\varepsilon$ with "$-$" sign. The poles in question can occur at $Y_0(x)$, $Y_1(x)$. (The residue at infinity is zero.) If $x$ with $|x| = 1-\varepsilon$, is such that $|Y_0(x)| < 1-\varepsilon$, we have only the residue at $Y_1(x)$.

Let us fix now $x$ with $|x| = 1-\varepsilon$, such that $|Y_0(x)| \ge 1-\varepsilon$. The numerator of (3.42) in a neighbourhood of $Y_0(x)$ is a holomorphic function of $y$, moreover $q(x, Y_0(x))\pi^{i_0 j_0}(x) + x^{i_0} Y_0^{j_0}(x) = 0$. In fact, the point $s \in \mathbf{S}$, such that $x(s) = x$, $y(s) = Y_0(x)$, belongs to the domain $D_1$ on $\mathbf{S}$, where (3.61) holds and $\pi^{i_0 j_0}(x(s)) = \pi^{i_0 j_0}(s)$. Then the residue at $Y_0(x)$ is always zero.

Therefore the inner integral (3.41) equals the residue at $Y_1(x)$ with "$-$" sign for all $x$ with $|x| = 1-\varepsilon$. Hence, we get (3.44) and recall the definitions of $\Gamma_{1-\varepsilon}$ and $d\omega$. □

**Lemma 3.13.** *Let $i = r\cos(\gamma(r))$, $j = \sin(\gamma(r))$ and let $\gamma(r) \to \gamma$, as $r \to \infty$, where $\gamma \in [0, \gamma_E^*)$. Then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{q(1, p_{0-1}/p_{01}) \operatorname{res}_{x=1} q^{-1}(x, Y_0(x))}{p_{0-1} - p_{01}} \left(\frac{p_{01}}{p_{0-1}}\right)^j. \tag{3.65}$$

*Proof.* It is carried out analogously to the proof of Lemma 3.10. First, we state that the integral

$$\pi_{ij}^{i_0j_0} = \frac{1}{2\pi i}\int\limits_{\Gamma_{1-\varepsilon}} \frac{q(s)\pi^{i_0j_0}(s)+x^{i_0}(s)y^{j_0}(s)}{\big(x^{\operatorname{ctg}\gamma}(s)y(s)\big)^j}\,d\omega, \tag{3.66}$$

equals the integral along the shifted contour $\Gamma_\gamma$ summed over the residues of the integrand at the poles in $E_\gamma$ as in (3.48).

Let us choose $\varepsilon > 0$, such that there are no poles of $\pi^{i_0j_0}(x)$ in $1-\varepsilon < |x| < 1$. Next, Proposition 3.3 is proved in this case exactly as in Lemma 3.9. (The domain $h_xE_\gamma$ being outside $|x| = 1-\varepsilon$ on $\mathbf{C}_x$, the poles of $h_x\pi^{i_0j_0}(s)$ are at the poles of $\sum_{i=0}^{\infty}\pi_{i0}^{i_0j_0}x^i$. We show that the images of the level curves $h_x\Gamma(\gamma, s^*)$ are outside the circles $|x| = x(s^*)$ for $s^* \in [s_1, s_E^*-\varepsilon]$ and apply the Hadamard–Pringsheim theorem.) This implies that the asymptotics of (3.48) is determined by the "lowest" pole on $(s(\gamma), s_E^* - \varepsilon)$, whenever it exists, and by the saddle-point (3.49) otherwise.

Remark 3.5 ensures that $\pi^{i_0j_0}(s)$ has a pole at $s_E^* = (1, p_{0-1}/p_{01})$. By structural stability and Proposition 3.3, it is the "lowest" one for all given $\gamma \in [0, \gamma_E^*)$. Therefore, the asymptotics of the integral (3.66) is determined by it and is uniform in a neighbourhood of $\gamma$. Thus by Lemma 3.12, we have

$$\pi_{ij}^{i_0j_0} \sim \frac{q(1,\,p_{0-1}/p_{01})\operatorname{res}_{x=1}\pi^{i_0j_0}(x)}{2a(1)p_{0-1}/p_{01}+b(1)}\Big(\frac{p_{01}}{p_{0-1}}\Big)^j.$$

It remains to notice that $\pi^{i_0j_0}(x) = -x^{i_0}Y_0^{j_0}(x)q^{-1}(x, Y_0(x))$ in a neigbourhood of $x = 1$, where $Y_0(1) = 1$. □

**Lemma 3.14.** *Let $i = r\cos(\gamma(r))$, $j = \sin(\gamma(r))$, and let $\gamma(r) \to \gamma$, as $r \to \infty$, where $\gamma \in (\gamma_E^*, \pi]$.*
(a) *Assume that the function $\pi^{i_0j_0}(s)$ has no poles on the interval $(s_E^*, s(\gamma))$, $\gamma \neq \pi$. Then*

$$\pi_{ij}^{i_0j_0} \sim \frac{C(\gamma, i_0, j_0)}{\sqrt{j}\,x^i(\gamma(r))y^j(\gamma(r))} \qquad \text{as } r \to \infty, \tag{3.67}$$

*where*

$$\begin{aligned} C(\gamma, i_0, j_0) = {}& \big[q(x(\gamma), y(\gamma))\pi^{i_0j_0}(x(\gamma)) + x^{i_0}(\gamma)y^{j_0}(\gamma)\big]\big|x^{\operatorname{ctg}\gamma}(\gamma)y(\gamma)\big|^{1/2} \\ & \times \big[2a(x(\gamma))y(\gamma)+b(x(\gamma))\big]^{-1}\Big|\frac{d^2\,x^{\operatorname{ctg}\gamma}(\gamma)Y(x(\gamma))}{dx^2}\Big|^{-1/2}. \end{aligned} \tag{3.68}$$

*If $\gamma = \pi$, $q(s_3) \neq 0$ and the function $\pi^{i_0j_0}(s)$ has no poles on the interval $(s_E^*, s_3)$, then*

$$\pi_{ij}^{i_0j_0} \sim \frac{1}{x^i(\gamma(r))y^j(\gamma(r))}\Big(\frac{\widetilde{C}(\gamma(r), i_0, j_0)}{\sqrt{|i|}} + \frac{C_2(\gamma(r), i_0j_0)}{|i|\sqrt{|i|}}\Big), \tag{3.69}$$

*and*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = \lim_{\gamma\to\pi} \frac{C(\gamma,i_0,j_0)}{C(\gamma,0,0)}. \tag{3.70}$$

(b) *Assume that the function* $\pi^{i_0j_0}(s)$ *has exactly one pole* $s'$ *on* $(s^*_{\scriptscriptstyle E}, s(\gamma))$, $q(s')\neq 0$, $\mathrm{res}_{x(s')}\,\pi^{i_0j_0}(x)\neq 0$. *Then*

$$\pi^{i_0j_0}_{ij} \sim \frac{q(x(s'),y(s'))\,\mathrm{res}_{x(s')}\,\pi^{i_0j_0}(x)}{x^i(s')y^j(s')[2a(x(s'))y(s')+b(x(s'))]}. \tag{3.71}$$

*Proof.* As usual we begin by finding the asymptotics of the integral (3.66). To get (3.48), we shift the contour to the saddle-point $s(\gamma)$, as in Lemma 3.9. Next, one can prove Proposition 3.3. (The domain $h_xE_\gamma$ lies inside $|x|=1-\varepsilon$, the poles of $h_x\pi^{i_0j_0}(s)x^i$ are at the poles of $\sum_{i=-\infty}^{0}\pi^{i_0j_0}_{i0}x^i$. One can show that the level curves $h_x\Gamma(\gamma,s^*)$ are outside the circle $|x|=x(s^*)$.)

It follows from this proposition that in case (a) of the theorem there are no poles of $\pi^{i_0j_0}(s)$ in $E_\gamma$ "lower" than the saddle-point or at its level. Then the asymptotics of (3.66) is determined by the contribution of the saddle-point. If $(i_0,j_0)$ is such that $C(\gamma,i_0,j_0)\neq 0$, then similarly to Lemma 3.9 we have (3.68). As in Proposition 3.4 one can get that in fact $C(\gamma,i_0,j_0)\neq 0$ for all pairs $(i_0,j_0)$, $\gamma\neq\pi$. The case $\gamma=\pi$ is treated analogously to Proposition 3.5.

In case (b), the pole $s'$ is the "lowest" pole in $E_\gamma$ and the asymptotics of (3.48) is determined by it. Similarly to Lemma 3.10 we obtain (3.71). □

**Lemma 3.15.** *The function* $\pi^{i_0j_0}(s)$ *has a pole in* $s'\in(s^*_{\scriptscriptstyle E},s_3)$ *if and only if* $q(\xi s')=0$. *This holds if and only if* $q(s_3)>0$. *This pole on the interval* $(s^*_{\scriptscriptstyle E},s_3)$ *is unique. Moreover* $q(s')\neq 0$, $\mathrm{res}_{x(s')}\,\pi^{i_0j_0}(x)\neq 0$.

*Proof.* In accordance with the definition of the function $\pi^{i_0j_0}(s)$ on the interval $(s^*_{\scriptscriptstyle E},s_3)$ belonging to $D_2$, we have

$$\begin{aligned}\pi^{i_0j_0}(s)=\pi(\xi s) &= -\frac{x^{i_0}(s)y^{j_0}(\xi s)}{q(\xi s)}\\ &= -\frac{x^{i_0}(s)\big(p_{0-1}/(p_{01}y(s))\big)^{j_0}}{q\big(x(s),\ p_{0-1}/(p_{01}y(s))\big)}.\end{aligned}$$

Thus $s'\in(s^*_{\scriptscriptstyle E},s_3)$ is a pole of $\pi^{i_0j_0}(s)$ if and only if $\xi s\in(s_3\,,\,(1,1))$ is a zero of $q(s)$. Then the result follows from Lemma 2.6. It is shown in the proof of this lemma that the zero is of the first order and so $\mathrm{res}_{x(s')}\,\pi^{i_0j_0}(x)\neq 0$. Moreover if $q(\xi s)=0$, $s\neq s_1,s_3$, $p'_{-11}+p'_{01}+p'_{11}\neq 0$, then $q(s)\neq 0$ as in Lemma 3.11. □

**Proposition 3.7.** *Let* $i=r\cos(\gamma(r))$, $j=r\sin(\gamma(r))$, $\gamma(r)\to\gamma$ *as* $r\to\infty$.

*Assume that the function* $\pi^{i_0j_0}(s)$ *has a pole in* $s(\gamma)\in(s^*_{\scriptscriptstyle E},s_3)$ *and no poles on* $(s^*_{\scriptscriptstyle E},s(\gamma))$. *Then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = x^{i_0}(\gamma)\left(p_{0-1}/p_{01}y(\gamma)\right)^{j_0}. \tag{3.72}$$

*If* $\gamma = \gamma^*_E$*, then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) = 1. \tag{3.73}$$

*Proof.* It is similar to the proof of Proposition 3.6. □

*Proof of Theorem* 2.3. If $\gamma \in [0,\gamma^*_E)$, Lemma 3.13 applies. By the definition of the Martin kernel (2.28) holds.

Assume that $q(s_3) < 0$. By Lemma 3.15 $\pi^{i_0j_0}(s)$ has no poles on $(s^*_E, s_3)$. Then for all $\gamma \in (\gamma^*_E,\pi]$ we can substitute the asymptotics of $\pi^{i_0j_0}_{ij}$ found in case (a) of Lemma 3.14 into the definition of the Martin kernel. The definition (3.63) of the function $\pi^{i_0j_0}(s)$ entails (2.29).

Assume that $q(s_3) > 0$. By Lemma 3.15 there is exactly one pole $s'$ on $(s^*_E, s_3)$, $q(\xi s') = 0$. According to the notations of Subsection 2.4 the angle $\gamma'$ is such that $s' = s(\gamma')$; $x' = x(s')$, $y' = y(\xi s') = p_{0-1}/(p_{01}y(s'))$, $q(x',y') = 0$. To find the asymptotics of the Martin kernel we are entitled to use the case (a) of Lemma 3.14 for $\gamma \in (\gamma^*_E, \gamma')$ and case (b) for $\gamma \in (\gamma',\pi)$. This gives (2.29) and (2.30) respectively. For $\gamma = \gamma^*_E, \gamma'$ we apply Proposition 3.7. The proof of the theorem is established. □

### 3.5. Random walk in $\mathbf{Z}^2_+$, $\mathsf{E}_x > 0$, $\mathsf{E}_y > 0$: proofs

*Proof of Lemma* 2.7. This lemma is equivalent to the following statement: *The function* $q(s)$ [*resp.* $\widetilde{q}(s)$] *has a zero on the interval* $((1, p_{0-1}/p_{01})\ s_1)$ [*resp.* $(s_2\ ,\ (p_{-10}/p_{01}, 1))$] *if and only if* $q(s_1) > 0$ [*resp.* $\widetilde{q}(s_2) > 0$]. *This zero on the corresponding segment is unique.* Moreover we show that *this zero is of the first order.*

This statement for the function $q(s)$ has already been proved in Lemma 2.5. The proof is completely the same for the function $\widetilde{q}(s)$ if one exchanges the roles of $x$ and $y$. □

**Lemma 3.16.** *If* $|x| < 1$, $|y| < 1$, $|z| \le 1$, *the following equation holds*

$$\begin{aligned} &Q(x,y,z)\sum_{i,j=1}^{\infty} \pi^{i_0j_0}_{ij}(z)x^{i-1}y^{j-1} \\ &= \quad q(x,y,z)\pi^{i_0j_0}(x,z) + \widetilde{q}(x,y,z)\widetilde{\pi}^{i_0j_0}(y,z) + q_0(x,y,z)\pi^{i_0j_0}_{00}(z) + x^{i_0}y^{j_0}, \end{aligned} \tag{3.74}$$

*where*

$$Q(x,y,z) = xy\big(1 - z(p_{10}x + p_{01}y + p_{-10}x^{-1} + p_{0-1}y^{-1})\big),$$

$$q(x,y,z) = x\Big(z\sum_{i,j} p'_{ij}x^iy^j - 1\Big), \qquad \widetilde{q}(x,y,z) = y\Big(z\sum_{i,j} p''_{ij}x^iy^j - 1\Big),$$

$$q_0(x,y,z) = z\sum_{i,j} p^0_{ij}x^iy^j - 1,$$

$$\pi^{i_0j_0}(x,z) = \sum_{i=1}^{\infty} \pi_{i0}^{i_0j_0}(z)x^{i-1}, \qquad \pi^{i_0j_0}(x,z) = \sum_{j=1}^{\infty} \pi_{0j}^{i_0j_0}(z)y^{i-1}.$$

*Proof.* We have

$$P_{ij}^{i_0j_0}(t+1)$$
$$= \sum_{k,l=1}^{\infty} p_{i-kj-l}P_{kl}^{i_0j_0}(t) + \sum_{k=1}^{\infty} p'_{i-kj}P_{k0}^{i_0j_0}(t) + \sum_{l=1}^{\infty} p''_{ij-l}P_{0l}^{i_0j_0}(t) + p^{0}_{ij}P_{00}^{i_0j_0}(t).$$

This equation together with the definition of $\pi_{ij}^{i_0j_0}(z)$ implies

$$\begin{aligned}
&\pi_{ij}^{i_0j_0}(z) - \pi_{ij}^{i_0j_0}(0) \\
&\quad= z\Big(\sum_{k,l=1}^{\infty} p_{i-kj-l}\pi_{kl}^{i_0j_0}(z) + \sum_{k=1}^{\infty} p'_{i-kj}\pi_{k0}^{i_0j_0}(z) \\
&\qquad + \sum_{l=1}^{\infty} p''_{ij-l}\pi_{0l}^{i_0j_0}(z) + p^{0}_{ij}P_{00}^{i_0j_0}(z)\Big) \quad \text{for all } i,j \geq 0,
\end{aligned}$$

where $|z| \leq 1$. Let us multiply this equation by $x^i y^j$, $|y| < 1$, $|x| < 1$. Taking the summation over $i, j$, and changing the order of the summation, we have

$$\begin{aligned}
&\sum_{i,j=1}^{\infty} \pi_{ij}^{i_0j_0}(z)x^i y^j + \sum_{i=1}^{\infty} \pi_{i0}^{i_0j_0}(z)x^i + \sum_{j=1}^{\infty} \pi_{0j}^{i_0j_0}(z)y^j + \pi_{00}^{i_0j_0}(z) - x^{i_0}y^{j_0} \\
&\quad= z\sum_{i,j} p_{ij}x^i y^j \sum_{i,j=1}^{\infty} \pi_{ij}^{i_0j_0}(z)x^i y^j + z\sum_{i,j} p'_{ij}x^i y^j \sum_{i=1}^{\infty} \pi_{i0}^{i_0j_0}(z)x^i \\
&\qquad + z\sum_{i,j} p''_{ij}x^i y^j \sum_{j=1}^{\infty} \pi_{0j}^{i_0j_0}(z)y^j + z\sum_{i,j} p^{0}_{ij}x^i y^j \pi_{00}^{i_0j_0}(z).
\end{aligned}$$

By simple probabilistic arguments $\pi_{ij}^{i_0j_0}(1) \leq C$, then the sums in the last equation are finite and we get (3.74). □

**Corollary 3.3.** *If $|x| < 1$, $|y| < 1$ the following equation holds*

$$\begin{aligned}
&Q(x,y)\sum_{i,j=1}^{\infty} \pi_{ij}^{i_0j_0} x^{i-1}y^{j-1} \\
&\quad= q(x,y)\pi^{i_0j_0}(x) + \widetilde{q}(x,y)\widetilde{\pi}^{i_0j_0}(y) + q_0(x,y)\pi_{00}^{i_0j_0} + x^{i_0}y^{j_0},
\end{aligned} \tag{3.75}$$

*where*

$$\pi^{i_0j_0}(x) = \sum_{i=1}^{\infty} \pi_{i0}^{i_0j_0} x^{i-1}, \quad \widetilde{\pi}^{i_0j_0}(y) = \sum_{j=1}^{\infty} \pi_{0j}^{i_0j_0} x^{j-1}.$$

*Proof.* This is equation (3.74) with $z = 1$. In agreement with the notations (2.1) and (2.31) we have $Q(x,y,1) = Q(x,y)$, $q(x,y,1) = q(x,y)$, $\widetilde{q}(x,y,1) = \widetilde{q}(x,y)$, $q_0(x,y,1) = q_0(x,y)$. □

In the domain $\Delta_0 = \{s : |x(s)| < 1, |y(s)| < 1\}$ of the Riemann surface $\mathbf{S}$ we have

$$q(s)\pi^{i_0 j_0}(x(s)) + \widetilde{q}(s)\widetilde{\pi}^{i_0 j_0}(y(s)) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s)y^{j_0}(s) = 0. \qquad (3.76)$$

We will define now the functions $\pi^{i_0 j_0}(s)$ and $\widetilde{\pi}^{i_0 j_0}(s)$ on $\mathbf{S}$ relying using this equation.

*Definition of the functions $\pi^{i_0 j_0}(s)$ and $\widetilde{\pi}^{i_0 j_0}(s)$ on* $\mathbf{S}$.

Let us divide the Riemann surface into four domains: $\Delta_0$, $\Delta_1$, $\Delta_2$, $\Delta_3$. The domain $\Delta_1$ [resp. $\Delta_2$] is bounded by $\Gamma_0$ [resp. $\widetilde{\Gamma}_0$] and the curve $\{s : x_3 \le x(s) \le x_4\}$ [resp. $\{s : y_3 \le y(s) \le y_4\}$] and it contains the interval $((1, p_{0-1}/p_{01}), s_1)$ [resp. $(s_2, (p_{-10}/p_{01}, 1))$] of $F_0$. The domain $\Delta_3$ is bounded by the curves $\{s : x_3 \le x(s) \le x_4\}$, $\{s : y_3 \le y(s) \le y_4\}$ and it contains the interval $(s_1, s_2) \subset F_0$.

On $\Delta_0$ we put:

$$\begin{aligned} \pi^{i_0 j_0}(s) &:= \pi^{i_0 j_0}(x(s)) = \sum_{i=1}^{\infty} \pi_{i0}^{i_0 j_0} x^{i-1}(s), \qquad s \in \Delta_0, \\ \widetilde{\pi}^{i_0 j_0}(s) &:= \widetilde{\pi}^{i_0 j_0}(y(s)) = \sum_{j=1}^{\infty} \pi_{0j}^{i_0 j_0} y^{j-1}(s). \end{aligned} \qquad (3.77)$$

In the domain $\overline{\Delta_1}$ we have $|y(s)| < 1$ and we put

$$\begin{aligned} \widetilde{\pi}^{i_0 j_0}(s) &:= \widetilde{\pi}^{i_0 j_0}(y(s)) = \sum_{j=0}^{\infty} \pi_{0j}^{i_0 j_0} y^{j-1}(s), \quad s \in \overline{\Delta_1}, \\ \pi^{i_0 j_0}(s) &:= -\frac{\widetilde{q}(s)\widetilde{\pi}^{i_0 j_0}(s) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s)y^{j_0}(s)}{q(s)}. \end{aligned} \qquad (3.78)$$

In the domain $\overline{\Delta_2}$ we have $|x(s)| < 1$ and we put

$$\begin{aligned} \pi^{i_0 j_0}(s) &:= \pi^{i_0 j_0}(x(s)) = \sum_{i=0}^{\infty} \pi_{i0}^{i_0 j_0} x^{i-1}(s), \qquad s \in \overline{\Delta_2}, \\ \widetilde{\pi}^{i_0 j_0}(s) &:= -\frac{q(s)\pi^{i_0 j_0}(s) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s)y^{j_0}(s)}{\widetilde{q}(s)}. \end{aligned} \qquad (3.79)$$

In order to define these functions in $\Delta_3$, we find for all $s \in \Delta_3$ the points $\xi s \in \Delta_0 \cup \Delta_1$ and $\eta s \in \Delta_0 \cup \Delta_2$, where $\xi$ and $\eta$ are the Galois automorphisms (3.6). Let us put

$$\pi^{i_0 j_0}(s) := \pi^{i_0 j_0}(\xi s), \qquad \widetilde{\pi}^{i_0 j_0}(s) := \widetilde{\pi}^{i_0 j_0}(\eta s), \qquad s \in \Delta_3. \qquad (3.80)$$

Thus the functions $\pi^{i_0 j_0}(s)$ and $\widetilde{\pi}^{i_0 j_0}(s)$ are defined on all of $\mathbf{S}$. Equation (3.76) holds in $\Delta_0 \cup \overline{\Delta_1} \cup \overline{\Delta_2}$, but generally does not hold in $\Delta_3$.

It is worthwhile to make some remarks.

*Remark* 3.6. The function $\pi^{i_0j_0}(s)$ is *meromorphic* on $\mathbf{S}$ cut along $\{s : x_3 \leq x(s) \leq x_4\}$. The function $\widetilde{\pi}^{i_0j_0}(s)$ is *meromorphic* on $\mathbf{S}$ cut along $\{s : y_3 \leq y(s) \leq y_4\}$.

*Remark* 3.7. For all $s \in \mathbf{S}$, $\pi^{i_0j_0}(s) = \pi^{i_0j_0}(\xi s)$ and $\widetilde{\pi}^{i_0j_0}(s) = \widetilde{\pi}^{i_0j_0}(\eta s)$.

*Remark* 3.8. Consider the subdomain of $\Delta_3$, where $|y(s)| < 1$. It is bounded by $\{s : x_3 \leq x(s) \leq x_4\}$ and $\widetilde{\Gamma}_1$. Namely, it contains the interval $(s_1, s_E)$. For all $s$ of this subdomain, $\eta s \in \Delta_0$ and $\xi s \in \Delta_1$, thus

$$\widetilde{\pi}^{i_0j_0}(s) = \widetilde{\pi}^{i_0j_0}(\eta s) = \widetilde{\pi}^{i_0j_0}(y(\eta s)) = \sum_{j=0}^{\infty} \pi_{0j}^{i_0j_0} y^{j-1}(s), \tag{3.81}$$

$$\pi^{i_0j_0}(s) = \pi^{i_0j_0}(\xi s) = -\frac{\widetilde{q}(\xi s)\widetilde{\pi}^{i_0j_0}(\xi s) + q_0(\xi s)\pi_{00}^{i_0j_0} + x^{i_0}(s)y^{j_0}(\xi s)}{q(\xi s)},$$

where $y(\xi s) = p_{0-1}/(p_{01}y(s)) < 1$.

Consider the subdomain of $\Delta_3$, where $|x(s)| < 1$. It is bounded by $\{s : y_3 \leq x(s) \leq y_4\}$ and $\Gamma_1$. In particular, it contains the interval $(s_E, s_2)$. For all $s$ of this subdomain, $\xi s \in \Delta_0$ and $\eta s \in \Delta_2$, thus

$$\pi^{i_0j_0}(s) = \pi^{i_0j_0}(\xi s) = \pi^{i_0j_0}(x(\xi s)) = \sum_{i=0}^{\infty} \pi_{i0}^{i_0j_0} x^{i-1}(s), \tag{3.82}$$

$$\widetilde{\pi}^{i_0j_0}(s) = \widetilde{\pi}^{i_0j_0}(\eta s) = -\frac{q(\eta s)\pi^{i_0j_0}(\eta s) + q_0(\eta s)\pi_{00}^{i_0j_0} + x^{i_0}(\eta s)y^{j_0}(s)}{q(\eta s)},$$

where $x(\eta s) = p_{-10}/(p_{10}x(s)) < 1$.

*Remark* 3.9. The functions $\pi^{i_0j_0}(s)$ and $\widetilde{\pi}^{i_0j_0}(s)$ have no pole in $s_E = (1, 1)$, since $q(1, p_{0-1}/p_{01}) < \infty$, $\widetilde{q}(p_{-10}/p_{01}, 1) < \infty$.

*Meromorphic continuation of the functions $\pi^{i_0j_0}(x)$ and $\widetilde{\pi}^{i_0j_0}(y)$ on $\mathbf{C}$.*

The functions $\pi^{i_0j_0}(x)$ and $\pi^{i_0j_0}(y)$ are defined and holomorphic in the domains $|x| < 1$ of $\mathbf{C}_x$ and $|y| < 1$ of $\mathbf{C}_y$ respectively. Setting

$$\begin{aligned} \pi^{i_0j_0}(x) &:= \pi^{i_0j_0}(s), \quad \text{where } s \text{ is such that } x(s) = x, \\ \widetilde{\pi}^{i_0j_0}(y) &:= \pi^{i_0j_0}(s), \quad \text{where } s \text{ is such that } y(s) = y, \end{aligned} \tag{3.83}$$

we obtain their meromorphic continuation on $\mathbf{C}_x$ cut along $[x_3, x_4]$ and on $\mathbf{C}_y$ cut along $[y_3, y_4]$ respectively. Since the functions $\pi^{i_0j_0}(s)$ and $\widetilde{\pi}^{i_0j_0}(s)$ have no pole in $s_E = (1, 1)$, the functions $\pi^{i_0j_0}(x)$ and $\widetilde{\pi}^{i_0j_0}(y)$ have no pole in $x = 1$ and $y = 1$ respectively. Proposition 2.1 is thus proved. (It can be also proved by purely probabilistic arguments.)

**Lemma 3.17.** *For all $j > j_0$ and all $i_0 \geq 0$*

$$
\begin{aligned}
\pi_{ij}^{i_0 j_0} = {} & \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{q(s)\pi^{i_0 j_0}(s)}{x^i(s)y^j(s)}\, d\omega + \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{q_0(s)\pi_{00}^{i_0 j_0}}{x^i(s)y^j(s)}\, d\omega \\
& + \frac{1}{2\pi i} \int\limits_{\widetilde{\Gamma}_1} \frac{\widetilde{q}(s)\widetilde{\pi}^{i_0 j_0}(s)}{x^i(s)y^j(s)}\, d\omega + \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{x^{i_0}(s)y^{j_0}(s)}{x^i(s)y^j(s)}\, d\omega. \qquad (3.84)
\end{aligned}
$$

*For all $i \geq i_0$ and $j_0 \geq 0$ (3.84) holds as well with $\widetilde{G}_1$ the contour in the last integral.*

*Proof.* The proof is similar to the proof of Lemma 3.8 taking into account equation (3.75). To get the integral along $\widetilde{\Gamma}_1$, one should exchange the roles of $x$ and $y$. □

**Lemma 3.18.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and let $\gamma(r) \to \gamma$ as $r \to \infty$, where $\gamma \in (0, \pi/2)$ and*

$$
\begin{aligned}
& q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(s(\gamma)) + \widetilde{q}(x(\gamma), y(\gamma))\widetilde{\pi}^{i_0 j_0}(s(\gamma)) \\
& \qquad + q_0(x(\gamma), y(\gamma))\pi_{00}^{i_0 j_0} + x^{i_0}(\gamma)y^{j_0}(\gamma) \quad \neq \quad 0.
\end{aligned}
$$

• *If $0 < \gamma \leq \gamma_E$, i.e. $s(\gamma) \in (s_1, s_E]$, assume that the function $\pi^{i_0 j_0}(s)$ has no poles on the segment $[s(\gamma), s_E]$. Then*

$$
\pi_{ij}^{i_0 j_0} \sim \frac{C(\gamma, i_0, j_0)}{\sqrt{j}\, x^i(\gamma(r))y^j(\gamma(r))} \qquad \text{as } r \to \infty, \qquad (3.85)
$$

*where*

$$
\begin{aligned}
C(\gamma, i_0, j_0) = {} & \big[q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(x(\gamma)) + \widetilde{q}(x(\gamma), y(\gamma))\widetilde{\pi}^{i_0 j_0}(y(\gamma)) \\
& + q_0(x(\gamma), y(\gamma))\pi_{00}^{i_0 j_0} + x^{i_0}(\gamma)y^{j_0}(\gamma)\big] \qquad (3.86) \\
& \times \left|x^{\operatorname{ctg}\gamma}(\gamma)y(\gamma)\right|^{1/2} \big[2a(x(\gamma))y(\gamma) + b(x(\gamma))\big]^{-1} \left|\frac{d^2\, x^{\operatorname{ctg}\gamma}(\gamma)Y(x(\gamma))}{dx^2}\right|^{-1/2}.
\end{aligned}
$$

• *If $\pi > \gamma \geq \gamma_E$, i.e. $s(\gamma) \in [s_E, s_2)$, assume that the function $\widetilde{\pi}^{i_0 j_0}(s)$ has no poles on the segment $[s_E, s(\gamma)]$. Then the asymptotics of $\pi_{ij}^{i_0 j_0}$ is given by (3.85) with the constant (3.86).*

*Proof.* The function $\widetilde{\pi}^{i_0 j_0}(s)$ [resp. $\pi^{i_0 j_0}(s)$] has no poles on $[s(\gamma), s_E]$ [resp. $[s_E, s(\gamma)]$] if $\gamma < \gamma_E$ [resp. $\gamma > \gamma_E$] by its definition (3.81) [resp. (3.82)]. Thus, under the assumptions of the theorem, all integrands in (3.84) have no poles on $[s(\gamma), s_E]$ [resp. $[s_E, s(\gamma)]$]. Taking into account the previous lemma, this proof can be carried out via the saddle-point method. It is quite similar to the one of Lemma 3.9 and details are omitted. □

**Proposition 3.8.** *If $0 < \gamma \le \gamma_E$, i.e. $s(\gamma) \in (s_1, s_E]$, assume that the function $\pi^{i_0 j_0}(s)$ has no poles on the segment $[s(\gamma), s_E)$. If $\pi > \gamma \ge \gamma_E$, i.e. $s(\gamma) \in [s_E, s_3)$, assume that the function $\widetilde{\pi}^{i_0 j_0}(s)$ has no poles on the segment $(s_E, s(\gamma)]$. Then for all pairs $(i_0, j_0)$*

$$q(x(\gamma), y(\gamma))\pi^{i_0 j_0}(s(\gamma)) + \widetilde{q}(x(\gamma), y(\gamma))\widetilde{\pi}^{i_0 j_0}(s(\gamma)) \\ + q_0(x(\gamma), y(\gamma))\pi_{00}^{i_0 j_0} + x^{i_0}(\gamma) y^{j_0}(\gamma) \quad \neq \quad 0. \tag{3.87}$$

*Proof.* Let us find a pair $(i_0, j_0)$ satisfying (3.87). Let e.g. $0 < \gamma < \gamma_E$, then $x(\gamma) > 1$, $y(\gamma) < 1$. Substitute into the left-hand side of (3.87) definition (3.81) of $\pi^{i_0 j_0}(s)$. It appears in this formula that the term

$$x^{i_0} y^{j_0} \Big( 1 - \frac{q(x(\gamma) y(\gamma))}{q(x(\gamma), \xi y(\gamma))} \Big( \frac{p_{0-1}}{p_{01} y^2(\gamma)} \Big)^{j_0} \Big),$$

where $y(\gamma) > \sqrt{p_{0-1}/p_{01}}$, can be made infinitely large by the choice of $(i_0, j_0)$, while the other terms are bounded, when $i_0, j_0 \to \infty$. Thus the required inequality holds.

To show (3.87) for all pairs $(i_0, j_0)$, one proceeds exactly as in Proposition 3.1. □

**Proposition 3.9.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$, $\gamma(r) \to \gamma$ as $r \to \infty$.*

- *Assume that $\gamma = 0$. If $q(s_1) \neq 0$ and the function $\pi^{i_0 j_0}(s)$ has no poles on the interval $(s_1, s_E)$, then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{1}{x(\gamma(r))^i y(\gamma(r))^j} \Big( \frac{\widetilde{C}(\gamma(r), i_0, j_0)}{\sqrt{|i|}} + \frac{C_2(\gamma(r), i_0 j_0)}{|i|\sqrt{|i|}} \Big) \tag{3.88}$$

*and*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = \lim_{\gamma \to 0} \frac{C(\gamma, i_0, j_0)}{C(\gamma, 0, 0)}.$$

- *Assume that $\gamma = \pi/2$. If $q(s_2) \neq 0$ and the function $\pi^{i_0 j_0}(s)$ has no poles on the interval $(s_E, s_2)$. Then the asymptotics of $\pi_{ij}^{i_0 j_0}$ is (3.88) and*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = \lim_{\gamma \to \pi/2} \frac{C(\gamma, i_0, j_0)}{C(\gamma, 0, 0)}.$$

*The constant $C(\gamma, i_0, j_0)$ is defined by Lemma 3.18.*

*Proof.* All the arguments are analogous to Proposition 3.5 and we skip them. □

**Lemma 3.19.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and let $\gamma(r) \to \gamma$ as $r \to \infty$; where $\gamma \in [0, \pi/2]$.*

- *Assume that $\gamma < \gamma_E$, i.e. $s(\gamma) \in [s_1, s_E)$, and the function $\pi^{i_0 j_0}(s)$ has exactly one pole $s'$ on the interval $(s(\gamma), s_E) \subset F_0$, $q(s') \neq 0$ and $\operatorname{res}_{x(s')} \pi^{i_0 j_0}(x) \neq 0$. Then*

$$\pi_{ij}^{i_0 j_0} \sim \frac{q(x(s'), y(s')) \operatorname{res}_{x(s')} \pi^{i_0 j_0}(x)}{x^i(s') y^j(s') [2a(x(s')) y(s') + b(x(s'))]}. \tag{3.89}$$

• *Assume that $\gamma > \gamma_E$, i.e. $s(\gamma) \in (s_E, s_2]$, and the function $\widetilde{\pi}^{i_0j_0}(s)$ has exactly one pole $s''$ on the interval $(s_E, s(\gamma)) \subset F_0$, $q(s'') \neq 0$, $\mathrm{res}_{x(s'')}\, \widetilde{\pi}^{i_0j_0}(y) \neq 0$. Then*

$$\pi_{ij}^{i_0j_0} \sim \frac{\widetilde{q}(x(s''), y(s''))\, \mathrm{res}_{y(s'')}\, \widetilde{\pi}^{i_0j_0}(y)}{x^i(s'')y^j(s'')[2\widetilde{a}(y(s''))x(s'') + \widetilde{b}(y(s''))]}. \tag{3.90}$$

*Proof.* Let $\gamma < \gamma_E$ [resp. $\gamma > \gamma_E$]. The function $\widetilde{\pi}^{i_0j_0}(s)$ [resp. $\pi^{i_0j_0}(s)$] has no poles in $[s(\gamma), s_E]$ [resp. $[s_E, s(\gamma)]$] by its definition (3.81) [resp. (3.82)]. Then the asymptotics of all integrals in (3.84) except for the one of $q(s)\pi^{i_0j_0}(s)$ [resp. $\widetilde{q}(s)\widetilde{\pi}^{i_0j_0}(s)$] is determined by the contribution of the saddle-point. The asymptotics of the integral of $q(s)\pi^{i_0j_0}(s)$ [resp. $\widetilde{q}(s)\pi^{i_0j_0}(s)$] is determined by the "lowest" pole. This pole is $s'$ [resp. $s''$]. Hence, proceeding along the same lines as in Lemma 3.10 we get the result. □

**Lemma 3.20.** *The function $\pi^{i_0j_0}(s)$ has a pole in $s' \in (s_1, s_E)$ if and only if $q(\xi s') = 0$. This holds if and only if $q(s_1) > 0$. This pole on the interval $(s_1, s_E)$ is unique.*

*The function $\widetilde{\pi}^{i_0j_0}(s)$ has a pole in $s'' \in (s_E, s_2)$ if and only if $\widetilde{q}(\eta s') = 0$. This holds if and only if $\widetilde{q}(s_2) > 0$. This pole on the interval $(s_E, s_2)$ is unique.*

*Moreover $q(s') \neq 0$, $\widetilde{q}(s'') \neq 0$, $\mathrm{res}_{x(s')}\, \pi^{i_0j_0}(x) \neq 0$, $\mathrm{res}_{y(s'')}\, \widetilde{\pi}^{i_0j_0}(y) \neq 0$.*

*Proof.* If $s' \in (s_1, s_E)$ [resp. $s'' \in (s_E, s_2)$] is a pole of $\pi^{i_0j_0}(s)$ [resp. $\widetilde{\pi}^{i_0j_0}(s)$] it follows from definition (3.81) [resp. (3.82)] that $q(\xi s') = 0$, [resp. $\widetilde{q}(\eta s_2) = 0$] and by Lemma 2.7 $q(s_1) > 0$ [resp. $\widetilde{q}(s_2) > 0$]. To get the inverse, one should show that in definition (3.81) [resp. (3.82)] the numerator is non-zero in $s'$ [resp. $s''$] for all pairs $(i_0, j_0)$. This is true with $(i_0, j_0)$ sufficiently large in view of the term $x^{i_0}(s')y^{j_0}(\xi s')$. If for some pair $(i_0'', j_0'')$ it is not true, the function $\pi^{i_0j_0}(s)$ has no poles on $(s_1, s_E)$ and the asymptotics of the mean number of visits to $(i, j)$ starting from $(i_0'', j_0'')$ is determined by the saddle-point as in Lemma 3.18. The same arguments as in Proposition 3.1 make this impossible. All other details of the proof are similar to Lemma 3.11 and we omit them. □

**Proposition 3.10.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$, $\gamma(r) \to \gamma$ as $r \to \infty$.*

*Assume that the function $\pi^{i_0j_0}(s)$ has a pole in $s(\gamma) \in (s_1, s_E)$ and no poles on $(s(\gamma), s_E)$. Then*

$$\lim_{r\to\infty} \mathsf{k}_{ij}(i_0, j_0) = x^{i_0}(\gamma)\big(p_{0-1}/p_{01}y(\gamma)\big)^{j_0}. \tag{3.91}$$

*The same is true if $s(\gamma) \in (s_E, s_2)$ and the function $\widetilde{\pi}^{i_0j_0}(s)$ has no poles on $(s_E, s(\gamma))$.*

*Proof.* It is similar to Proposition 3.6. □

*Proof of Theorem* 2.4. 1. If $q(s_1) < 0$ and $\widetilde{q}(s_2) < 0$, then by Lemma 3.20 there are no poles of $\pi^{i_0j_0}(s)$ on $(s_1, s_E]$ and of $\widetilde{\pi}^{i_0,j_0}(s)$ on $[s_E, s_2)$. For all $\gamma \in (0, \pi/2)$

Lemma 3.18 applies. Then

$$\begin{aligned}
&\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0)\\
&\quad= \big[q(x(\gamma),y(\gamma))\pi^{i_0j_0}(s(\gamma)) + \widetilde{q}(x(\gamma),y(\gamma))\widetilde{\pi}^{i_0j_0}(s(\gamma))\\
&\qquad\quad + q_0(x(\gamma),y(\gamma))\pi_{00}^{i_0j_0} + x^{i_0}(\gamma)y^{j_0}(\gamma)\big]\\
&\qquad \times\big[q(x(\gamma),y(\gamma))\pi^{00}(s(\gamma)) + \widetilde{q}(x(\gamma),y(\gamma))\widetilde{\pi}^{00}(s(\gamma))\\
&\qquad\quad + q_0(x(\gamma),y(\gamma))\pi_{00}^{00} + 1\big]^{-1}.
\end{aligned}$$

If $\gamma \le \gamma_E$ one should substitute here definition (3.81) of $\pi^{i_0j_0}(s)$ on $(s_1, s_E]$; if $\gamma \ge \gamma_E$ one should substitute here definition (3.82) of $\widetilde{\pi}^{i_0j_0}(s)$ on $[s_E, s_2)$. Then (2.39) and (2.40) hold. For $\gamma = 0, \pi/2$ Proposition 3.9 applies.

2. If $q(s_1) > 0$ and $q(s_2) < 0$, by Lemma 3.20 there is exactly one pole $s'$ of the function $\pi^{i_0j_0}(s)$ on $(s_1, s_E)$, $q(\xi s') = 0$. The angle $\gamma'$ is such that $s(\gamma') = s'$. Then for $\gamma \in [0, s(\gamma'))$ Lemma 3.19 applies. In view of definition (3.83), $\pi^{i_0j_0}(x) = \pi^{i_0j_0}(s) = \pi^{i_0j_0}(\xi s)$, where $x(s) = x$. If $s' \in (s_1, s_E)$, then $\pi^{i_0j_0}(\xi s')$ can be found by (3.81). Hence

$$\begin{aligned}
\lim_{r\to\infty} \mathsf{k}_{ij}(i_0,j_0) &= \frac{\operatorname{res}_{x(s')} \pi^{i_0j_0}(x)}{\operatorname{res}_{x(s')} \pi^{00}(x)}\\
&= \frac{(x')^{i_0}(y')^{j_0} + q_0(x',y')\pi_{00}^{i_0j_0} + \widetilde{q}(x',y')\widetilde{\pi}^{i_0j_0}(y')}{1 + q_0(x',y')\pi_{00}^{00} + \widetilde{q}(x',y')\widetilde{\pi}^{00}(y')},
\end{aligned}$$

where $x' = x(s')$, $y' = y(\xi s') = p_{0-1}/(p_{01}y(s'))$.

The other details of the proof are similar to Theorem 2.2 taking into account the lemmas and propositions just proved. □

### 3.6. Random walk in $\mathbf{Z}_+^2$, $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$, escape to infinity along one axis: proofs

Let us mark some points on the Riemann surface. We have already introduced $s_E = (1,1) = \Gamma_0 \cap \widetilde{\Gamma}_0$ and $s_E^* = (1, p_{0-1}/p_{01}) = \Gamma_1 \cap F_0$. Let also $\widetilde{s}_E^* = (p_{-10}/p_{10}, 1) = \widetilde{\Gamma}_1 \cap F_0$ and $s_E^- = (p_{-10}/p_{10}, p_{0-1}/p_{01}) \subset (s_1, s_2) \in F_0$.

*Proof of Lemma* 2.8. It can be reformulated as follows: *the function* $\widetilde{q}(s)$ *has a zero on the interval* $(s_E^*, s_E) \subset F_0$ *if and only if* $\widetilde{q}(s_E^*) > 0$. Note that $\widetilde{q}_y(s_E) = \mathsf{E}_x^{-1}(\mathsf{E}_x\,\mathsf{E}_y'' - \mathsf{E}_y\,\mathsf{E}_x'') < 0$. The other details are similar to Lemmas 2.5 or 2.6. One shows in the proof that *this zero is of the first order.* □

Equation (3.75) remains valid, provided that $|x|$, $|y| < 1$.

We also need the following proposition.

**Proposition 3.11.** *We have*

$$\sum_{j=0}^{\infty} \pi_{0j}^{i_0j_0} < \infty.$$

*Proof.* We should prove the fact that the mean number of visits to the $y$-axis $S''$ is finite.

First, we show that the probability of reaching the $x$-axis $S'$ starting from $S''$ is 1. For this purpose, we construct the Lyapounov function as in case (ai) of Theorem 3.3.1 in [3] given by $f_1(x,y) = \sqrt{ux^2+vy^2+wxy}$, i.e. satisfying Theorem 2.2.3 (Foster's criterion) in [3] but where the set $A = S'$ is not finite (see also Figure 3.3.1 in [3]). Proceeding along the same lines as in the proof of this theorem, we derive the finiteness of the mean time to reach $S'$, starting from $S \cup S''$.

The next step is to show that the probability to never reach $S''$ starting from any point of $S'$ is greater than some $\delta > 0$. This is done by means of a Lyapounov function as in the case a(i) of Theorem 3.3.2 in [3], that is $f_2(x,y) = x - y/\varepsilon$, where $\mathsf{E}_y \,/\, \mathsf{E}_x < \varepsilon < \mathsf{E}_y^{'} \,/\, \mathsf{E}_x^{'}$ (see also Figure 3.3.3 in [3]). Thus we have the assumptions of Theorem 2.1.9 in [3] with $N_i = i$, $S_n = f(X_n, Y_n)$, $S_0 > 1$. In view of this theorem the probability to never reach the set of states $\{(i,j) : f(i,j) < 1\}$, which contains $S''$, is strictly positive.

These two steps give the result. $\square$

<u>*Definition of the functions $\pi^{i_0 j_0}(s)$ and $\widetilde{\pi}^{i_0 j_0}(s)$ on $\mathbf{S}$.*</u>

Let us divide the Riemann surface into five domains: $\Delta_x = \{s : |x(s)| < 1\}$, $\Delta_y = \{s : |y(s)| < 1\}$, $\Delta_1$, $\Delta_2$, $\Delta_3$. The domain $\Delta_1$ is bounded by $\Gamma_1$ and the curve $\{s : y_3 \le y(s) \le y_4\}$ and contains the interval $(s_2, s_E^*)$; the domain $\Delta_2$ is bounded by $\widetilde{\Gamma}_1$ and the curve $\{s : x_3 \le x(s) \le x_4\}$ and contains the interval $(\widetilde{s}_E^*, s_1)$. The domain $\Delta_3$ is bounded by $\{s : x_3 \le x(s) \le x_4\}$, $\{s : y_3 \le y(s) \le y_4\}$ and contains $(s_1, s_2)$. Only $\Delta_x \cap \Delta_y \neq \emptyset$ and in $\Delta_x \cap \Delta_y$ the equation (3.76) holds. Let us put for $s \in \Delta_x$

$$
\begin{aligned}
\pi^{i_0 j_0}(s) &:= \sum_{i=0}^{\infty} \pi_{i0}^{i_0 j_0} x^{i-1}(s), \\
\widetilde{\pi}^{i_0 j_0}(s) &:= -\frac{q(s)\pi^{i_0 j_0}(s) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s) y^{j_0}(s)}{\widetilde{q}(s)},
\end{aligned}
\tag{3.92}
$$

and for $s \in \overline{\Delta}_y$

$$
\begin{aligned}
\widetilde{\pi}^{i_0 j_0}(s) &:= \sum_{j=0}^{\infty} \pi_{0j}^{i_0 j_0} y^{j-1}(s), \\
\pi^{i_0 j_0}(s) &:= -\frac{\widetilde{q}(s)\widetilde{\pi}^{i_0 j_0}(s) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s) y^{j_0}(s)}{q(s)}.
\end{aligned}
\tag{3.93}
$$

If $s \in \overline{\Delta}_1$, then $\xi s \in \overline{\Delta}_y$, where $\xi$ is the Galois (3.6) automorphism. The function $\pi^{i_0 j_0}(s)$ has already been defined on $\Delta_y$ by (3.93). Then let us put for $s \in \overline{\Delta}_1$

$$
\begin{aligned}
\pi^{i_0 j_0}(s) &:= \pi^{i_0 j_0}(\xi s), \\
\widetilde{\pi}^{i_0 j_0}(s) &:= -\frac{q(s)\pi^{i_0 j_0}(s) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s) y^{j_0}(s)}{\widetilde{q}(s)}.
\end{aligned}
\tag{3.94}
$$

If $s \in \overline{\Delta}_2$, then $\eta s \in \overline{\Delta}_x$, where $\eta$ is the Galois (3.6) automorphism. The function $\widetilde{\pi}^{i_0 j_0}(s)$ has already been defined on $\overline{\Delta}_x$ by (3.92). Then let us put for $s \in \overline{\Delta}_2$

$$\begin{aligned} \widetilde{\pi}^{i_0 j_0}(s) &:= \widetilde{\pi}^{i_0 j_0}(\eta s), \\ \pi^{i_0 j_0}(s) &:= -\frac{\widetilde{q}(s)\widetilde{\pi}^{i_0 j_0}(s) + q_0(s)\pi_{00}^{i_0 j_0} + x^{i_0}(s)y^{j_0}(s)}{q(s)}. \end{aligned} \tag{3.95}$$

If $s \in \Delta_3$, then $\xi s \in \Delta_y \cup \Delta_2$, $\eta s \in \Delta_x \cup \Delta_1$, where $\pi^{i_0 j_0}(s)$ and $\widetilde{\pi}^{i_0 j_0}(s)$ have already been defined. Let us put for $s \in \Delta_3$

$$\begin{aligned} \pi^{i_0 j_0}(s) &:= \widetilde{\pi}^{i_0 j_0}(\xi s), \\ \widetilde{\pi}^{i_0 j_0}(s) &:= \widetilde{\pi}^{i_0 j_0}(\eta s). \end{aligned} \tag{3.96}$$

It is worthwhile to make some remarks.

- The function $\pi^{i_0 j_0}(s)$ is meromorphic on $\mathbf{S}$ cut along $\{s : x_3 \le x(s) \le x_4\}$. The function $\widetilde{\pi}^{i_0 j_0}(s)$ is meromorphic on $\mathbf{S}$ cut along $\{s : y_3 \le y(s) \le y_4\}$.
- For all $s \in \mathbf{S}$ $\pi^{i_0 j_0}(s) = \pi^{i_0 j_0}(\xi s)$ and $\widetilde{\pi}^{i_0 j_0}(s) = \widetilde{\pi}^{i_0 j_0}(\eta s)$.
- Equation (3.76) holds on $\overline{\Delta}_x \cup \overline{\Delta}_y \cup \overline{\Delta}_1 \cup \overline{\Delta}_2$, but generally does not hold on $\Delta_3$.
- If $s \in (\widetilde{s}_E^*, s_E^-) \subset F_0$, then $\eta s \in \Delta_x$ and by the definition of $\widetilde{\pi}^{i_0 j_0}(s)$

$$\begin{aligned} \widetilde{\pi}^{i_0 j_0}(s) &= \widetilde{\pi}^{i_0 j_0}(\eta s) \\ &= -\frac{q(\eta s)\pi^{i_0 j_0}(\eta s) + q_0(\eta s)\pi_{00}^{i_0 j_0} + x^{i_0}(\eta s)y^{j_0}(s)}{\widetilde{q}(\eta s)}. \end{aligned} \tag{3.97}$$

- By Proposition 3.11 and definition (3.93), the function $\widetilde{\pi}^{i_0 j_0}(s)$ has no pole at the point $s_E = (1,1)$. Consequently, in view of equation (3.76) the function $\pi^{i_0 j_0}(s)$ has a pole in $s_E$, since $q(1,1) = 0$. Thus $\widetilde{\pi}^{i_0 j_0}(s)$ has no pole at $\widetilde{s}_E^*$ and $\pi^{i_0 j_0}(s)$ has a pole in $s_E^*$.

*Meromorphic continuation of the functions $\pi^{i_0 j_0}(x)$ and $\widetilde{\pi}^{i_0 j_0}(y)$ on $\mathbf{C}$.*

The functions $\pi^{i_0 j_0}(x)$ and $\widetilde{\pi}^{i_0 j_0}(y)$ are defined and holomorphic in the domains $|x| < 1$ of $\mathbf{C}_x$ and $|y| < 1$ of $\mathbf{C}_y$ respectively. Setting

$$\begin{aligned} \pi^{i_0 j_0}(x) &:= \pi^{i_0 j_0}(s), \quad \text{where } s \text{ is such that } x(s) = x, \\ \widetilde{\pi}^{i_0 j_0}(y) &:= \pi^{i_0 j_0}(s), \quad \text{where } s \text{ is such that } y(s) = y, \end{aligned} \tag{3.98}$$

we obtain their meromorphic continuation on the complex plane $\mathbf{C}_x$ cut along $[x_3, x_4]$ and on $\mathbf{C}_y$ cut along $[y_3, y_4]$ respectively. Since the function $\pi^{i_0 j_0}(s)$ has a pole in $s_E = (1,1)$, the function $\pi^{i_0 j_0}(x)$ has a pole in $x = 1$. (Clearly, the function $\widetilde{\pi}^{i_0 j_0}(y)$ is holomorphic in $|y| < 1+\varepsilon$, $\varepsilon > 0$, in view of Proposition 3.11.)

**Lemma 3.21.** *For all $j > j_0$ and all $i \geq 0$*

$$\pi_{ij}^{i_0j_0} = \frac{1}{2\pi i}\int\limits_{\Gamma_{1-\varepsilon}} \frac{q(s)\pi^{i_0j_0}(s)}{x^i(s)y^j(s)}\,d\omega + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{q_0(s)\pi_{00}^{i_0j_0}}{x^i(s)y^j(s)}\,d\omega$$
$$+ \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{x^{i_0}(s)y^{j_0}(s)}{x^i(s)y^j(s)}\,d\omega + \frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}_1} \frac{\widetilde{q}(s)\widetilde{\pi}^{i_0j_0}(s)}{x^i(s)y^j(s)}\,d\omega. \tag{3.99}$$

*If $i > i_0$, $j \geq 0$, then* (3.99) *holds as well, where the integral of $x^{i_0}(s)y^{j_0}(s)$ is along $\widetilde{\Gamma}_1$.*

The contour $\Gamma_{1-\varepsilon}$ has been already defined in Subsection 3.4.

*Proof.* The proof of this Lemma is similar to the proofs of Lemmas 3.5, 3.8 and 3.12. Let us outline the differences.

Equation (3.75) implies

$$\pi_{ij}^{i_0j_0} = \int\limits_{|x|=1-\varepsilon_1}\int\limits_{|y|=1-\varepsilon_2} \frac{q\pi^{i_0j_0} + \widetilde{q}\pi^{i_0j_0} + q_0\pi_{00}^{i_0j_0} + x^{i_0}y^{j_0}}{x^iy^j\,Q(x,y)}\,dy\,dx. \tag{3.100}$$

Whenever $x$, $|x| = 1-\varepsilon_1$, is fixed, the integrand of (3.100) is holomorphic in $1-\varepsilon_2 < |y| < 1+\varepsilon_2$. In fact, $\widetilde{\pi}^{i_0j_0}(y)$ is holomorphic by Proposition 3.11 and a zero of $Q(x,y)$ in this domain, if it exsits, can not be a pole of the integrand due to equation (3.76) in $\overline{\Delta}_x \cup \overline{\Delta}_y$. Thus one can shift the contour in (3.100) to $|y| = 1+\varepsilon_2$. Next, we split this integral into the sum of integrals of $q\pi^{i_0j_0}$, $\widetilde{q}\widetilde{\pi}^{i_0j_0}$ etc. To consider the first term, $\varepsilon_1$ is taken such that the zeros of $Q(x,y)$, where $|x| = 1-\varepsilon$, satisfy the inequalities $|Y_0(x)| < 1+\varepsilon_2$ and $|Y_1(x)| > 1+\varepsilon_2$. To treat the second one, we show that for all fixed $y$, $|y| = 1+\varepsilon_2$, $Q(x,y)$ has two zeros $|X_0(y)| < 1$ and $|X_1(y)| > 1+\varepsilon_2$. (This is done in the same way as (3.43) in Lemma 3.8.) The other details of the proof come from Lemmas 3.5, 3.8 or 3.12. $\square$

**Lemma 3.22.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and $\gamma(r) \to \gamma$ as $r \to \infty$, where $\gamma \in [0,\pi/2]$.*

*Assume that the function $\widetilde{\pi}^{i_0j_0}(s)$ has no poles on the interval $(\widetilde{s}_E^*, s_E^-)$. Then for $\gamma \neq \pi/2$*

$$\pi_{ij}^{i_0j_0} \sim \frac{q(1,p_{0-1}/p_{01})\operatorname{res}_{x=1} q^{-1}(x,Y_0(x))}{p_{0-1}-p_{01}}\Big(\frac{p_{01}}{p_{0-1}}\Big)^j. \tag{3.101}$$

*If $\widetilde{q}(p_{-10}/p_{10},p_{0-1}/p_{01}) \neq 0$, $\gamma = \pi/2$, then*

$$\pi_{ij}^{i_0j_0} \sim \frac{q(1,p_{0-1}/p_{01})\operatorname{res}_{x=1} q^{-1}(x,Y_0(x))}{p_{0-1}-p_{01}}\Big(\frac{p_{01}}{p_{0-1}}\Big)^j + C\Big(\frac{p_{10}}{p_{-10}}\Big)^i\Big(\frac{p_{01}}{p_{0-1}}\Big)^j, \tag{3.102}$$

*where*

$$C = \frac{\widetilde{q}(p_{-10}/p_{10},p_{0-1}/p_{01})q(1,p_{0-1}/p_{01})\operatorname{res}_{y=1} q^{-1}(X(y),y)}{\widetilde{q}(1,p_{0-1}/p_{01})[2\widetilde{a}(p_{0-1}/p_{01})p_{-10}/p_{10} + \widetilde{b}(p_{0-1}/p_{01})]}.$$

*Proof.* Let $0 \le \gamma < \pi/2$. The asymptotics of the integral of $q\pi^{i_0j_0}$ along $\Gamma_1$ in (3.100) is exactly (3.101). This is proved by means of the saddle-point method and taking into account the fact that $s^*_E = (1, p_{0-1}/p_{01})$ is a pole of $\pi^{i_0j_0}(s)$. Let us show that we can neglect the other terms of (3.100). The asymptotics of the integral of $\widetilde{q}\widetilde{\pi}^{i_0j_0}$ along $\widetilde{\Gamma}_1$ is determined by the saddle-point or by the "lowest" pole of this function on $(\widetilde{s}^*_E, s_2)$, if it exists. By assumption, this pole can only lie on $[s^-_E, s_2)$. Being on this interval, it is "higher" than $s^*_E$: since $x(s) > 1$, $y(s) \ge p_{0-1}/p_{01}$, then $x^{\operatorname{ctg}\gamma}(s)y(s) > p_{0-1}/p_{01}$ for all $0 < \gamma < \pi/2$. Obviously, the asymptotics of the integrals of $q_0(s)$ and $x^{i_0}(s)y^{j_0}(s)$ in (3.100) is determined by the saddle-point, which is always "higher" than $s^*_E$ as well.

If $\gamma = \pi/2$ and $\widetilde{q}(p_{-10}/p_{10}, p_{0-1}/p_{01}) \ne 0$, we should take into account the pole of the function $\pi^{i_0j_0}(s)$ at $s^-_E$. □

**Lemma 3.23.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and $\gamma(r) \to \gamma$ as $r \to \infty$, where $\gamma \in [0, \pi/2]$.*

*Assume that the function $\pi^{i_0j_0}(s)$ has exactly one pole $s'$ on the interval $(\widetilde{s}^*_E, s^-_E)$, $q(s') \ne 0$, $\operatorname{res}_{y(s')} \widetilde{\pi}^{i_0j_0}(y) \ne 0$. Let an angle $\gamma_0$ be such that*

$$x(s')^{\operatorname{ctg}\gamma_0} y(s') = \frac{p_{0-1}}{p_{01}}. \tag{3.103}$$

*Then for $\gamma < \gamma_0$*

$$\pi^{i_0j_0}_{ij} \sim \frac{q(1, p_{0-1}/p_{01}) \operatorname{res}_{x=1} q^{-1}(x, Y_0(x))}{p_{0-1} - p_{01}} \Big(\frac{p_{01}}{p_{0-1}}\Big)^j; \tag{3.104}$$

*for $\gamma > \gamma_0$*

$$\pi^{i_0j_0}_{ij} \sim \frac{\widetilde{q}(x(s'), y(s')) \operatorname{res}_{y=1} \widetilde{\pi}^{i_0j_0}(y')}{[2\widetilde{a}(y(s'))x' + \widetilde{b}(y(s'))]x^i(s')y^j(s')}; \tag{3.105}$$

*for $\gamma = \gamma_0$*

$$\begin{aligned}\pi^{i_0j_0}_{ij} \quad \sim \quad & \frac{q(1, p_{0-1}/p_{01}) \operatorname{res}_{x=1} q^{-1}(x, Y_0(x)}{p_{0-1} - p_{01}} \Big(\frac{p_{0-1}}{p_{01}}\Big)^j \\ & + \frac{\widetilde{q}(x(s'), y(s')) \operatorname{res}_{y=1} \widetilde{\pi}^{i_0j_0}(y')}{[2\widetilde{a}(y(s'))x' + \widetilde{b}(y(s'))]x^i(s')y^j(s')}. \end{aligned} \tag{3.106}$$

*Proof.* The asymptotics of the integral of $q\pi^{i_0j_0}$ in (3.99) is determined by the pole at $s^*_E$ as in (3.101). The asymptotics of the integral of $\widetilde{q}\widetilde{\pi}^{i_0j_0}$ is determined either by the saddle-point $s(\gamma)$ or by the pole $s'$. The angle $\gamma_0$ is such that $s^*_E$ and $s'$ are at the same level, i.e. these poles both contribute to the asymptotics of $\pi^{i_0j_0}_{ij}$. Whenever $\gamma < \gamma_0$ [resp. $\gamma > \gamma_0$], $s^*_E$ is "lower" [resp. "higher"] than $s'$. The result follows. □

**Lemma 3.24.** *The function $\widetilde{\pi}^{i_0j_0}(s)$ has a pole $s'$ on the interval $(\widetilde{s}^*_E, s^-_E)$ if and only if $\widetilde{q}(\eta s) = 0$. This holds if and only if $\widetilde{q}(1, p_{0-1}/p_{01}) > 0$. This pole is unique and $\widetilde{q}(s') \ne 0$, $\operatorname{res}_{y(s')} \widetilde{\pi}^{i_0j_0}(y) \ne 0$.*

*Proof.* The statement of the lemma follows from the definition (3.97) of $\widetilde{\pi}^{i_0j_0}(s)$ on the corresponding interval and Lemma 2.8. □

*Proof of Theorem* 2.5. 1. If $\widetilde{q}(1, p_{0-1}/p_{01}) < 0$, then by Lemma 3.24 there are no poles of $\widetilde{\pi}^{i_0j_0}(s)$ on the interval $(\widetilde{s}^*_E, s^-_E)$. Lemma 3.22 applies.

2. If $\widetilde{q}(1, p_{0-1}/p_{01}) > 0$, then there is exactly one pole $s'$ on $(\widetilde{s}^*_E, s^-_E)$. In accordance with notation of Lemma 2.8 $x(\eta s') = x'$, $y(\eta s') = y'$ and by Lemma 3.24 $q(x', y') = 0$. Lemma 3.23 applies. For $\gamma < \gamma_0$ the asymptotics of $\pi^{i_0j_0}_{ij}$ is given by (3.104), which entails (2.46). For $\gamma = \gamma_0$, $\gamma > \gamma_0$ we have (3.106) and (3.105) respectively with $x(s') = p_{-10}/(p_{10}x')$, $y(s') = y'$. By (3.97) and (3.98) the asymptotics of the Martin kernel is given by (2.47) and (2.48).

Whenever $\operatorname{ctg}\gamma_0$ is irrational, for all $p \in \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$ we can find sequences of integers $i_n$, $j_n$, such that: $i_n, j_n \to +\infty$ and $i_n - j_n \operatorname{ctg}\gamma_0 \to p$. Whenever $\operatorname{ctg}\gamma_0 = q_1/q_2$ is rational, $(q_1, q_2 \in \mathbf{N})$, the same is true for $p \in \{-\infty\} \cup \widetilde{\mathbf{Z}} \cup \{+\infty\}$, where $\widetilde{\mathbf{Z}} = \{m/q_2 : \ m \in \mathbf{Z}\}$. Taking different $p$ we will get different harmonic functions in the right-hand side of (2.48). In particular, in the cases $p = -\infty$ and $p = +\infty$ the result is the same as in (2.46) and (2.47) respectively.

If $p \neq \pm\infty$, the harmonic functions in the right-hand side of (2.48) are linear combinations of harmonic functions in the right-hand side of (2.46) and (2.47). Thus, all of them should be excluded from the minimal Martin boundary. □

## 3.7. Random walk in $\mathbf{Z}^2_+$, $\mathsf{E}_x < 0$, $\mathsf{E}_y < 0$, escape to infinity along two axes: proofs

*Proof of Lemma* 2.9. It is left to the reader. □

Equation (3.75) holds, provided that $|x| < 1$, $|y| < 1$. The definition of the functions $\pi^{i_0j_0}_{ij}(s)$ and $\widetilde{\pi}^{i_0j_0}_{ij}(s)$ on the Riemann surface are the same as in the previous subsection. The crucial difference is that these functions *both* have a pole at the point $s_E = (1, 1)$. Then $\pi^{i_0j_0}(s)$ [resp. $\widetilde{\pi}^{i_0j_0}(s)$] has a pole at $s^*_E$ [resp. $\widetilde{s}^*_E$]. The function $\pi^{i_0j_0}(x)$ [resp. $\pi^{i_0j_0}(y)$] is meromorphic on the complex plane cut along $[x_3, x_4]$ [resp. $[y_3, y_4]$] and has a pole at $x = 1$ [resp. $y = 1$].

**Lemma 3.25.** *For all $j > j_0$ and all $i \geq 0$*

$$\begin{aligned}\pi^{i_0j_0}_{ij} = {} & \frac{1}{2\pi i}\int\limits_{\Gamma_{1-\varepsilon}} \frac{q(s)\pi^{i_0j_0}(s)}{x^i(s)y^j(s)}\,d\omega + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{q_0(s)\pi^{i_0j_0}_{00}}{x^i(s)y^j(s)}\,d\omega \\ & + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{x^{i_0}(s)y^{j_0}(s)}{x^i(s)y^j(s)}\,d\omega + \frac{1}{2\pi i}\int\limits_{\widetilde{\Gamma}_{1+\varepsilon}} \frac{\widetilde{q}(s)\widetilde{\pi}^{i_0j_0}(s)}{x^i(s)y^j(s)}\,d\omega \\ & + \frac{\widetilde{q}(p_{-10}/p_{10}, 1)\widetilde{C}(i_0, j_0)}{p_{-10} - p_{10}}\Big(\frac{p_{10}}{p_{-10}}\Big)^i. \end{aligned} \tag{3.107}$$

*For all $i > i_0$ and $j \geq 0$* (3.107) *holds, where the integral of $x^{i_0}(s)y^{j_0}(s)$ is along $\widetilde{\Gamma}_1$. The constant $\widetilde{C}(i_0, j_0)$ is defined by Lemma* 2.9.

*Proof.* The proof is almost the same as of Lemma 3.21. The difference is that upon shifting the contour from $|y| = 1-\varepsilon$ to $|y| = 1+\varepsilon$ in (3.100), one should take into account the pole of the function $\widetilde{\pi}^{i_0j_0}(y)$ in $y=1$. The residue at this pole equals $-\widetilde{C}(i_0,j_0)$. Thus we will have

$$-\frac{C(i_0,j_0)}{2\pi i}\int\limits_{|x|=1-\varepsilon}\frac{\widetilde{q}(x,1)}{x^iQ(x,1)}\,dx = \frac{\widetilde{q}(p_{-10}/p_{10},1)\widetilde{C}(i_0,j_0)}{p_{-10}-p_{10}}\Big(\frac{p_{10}}{p_{-10}}\Big)^i.$$

The other details are similar to Lemma 3.21. □

**Lemma 3.26.** *Let $i = r\cos(\gamma(r))$, $j = r\sin(\gamma(r))$ and $\gamma(r)\to\gamma$ as $r\to\infty$, where $\gamma\in[0,\pi/2]$. Let us define an angle $\gamma_0$ such that*

$$(p_{0-1}/p_{01})^{\operatorname{ctg}\gamma_0} = (p_{0-1}/p_{01}).$$

*Then for $\gamma<\gamma_0$*

$$\pi_{ij}^{i_0j_0}\sim\frac{q(p_{0-1}/p_{01},1)C(i_0,j_0)}{p_{0-1}-p_{01}}\Big(\frac{p_{01}}{p_{0-1}}\Big)^j;$$

*for $\gamma=\gamma_0$*

$$\pi_{ij}^{i_0j_0}\sim\frac{\widetilde{q}(p_{-10}/p_{10},1)\widetilde{C}(i_0,j_0)}{p_{-10}-p_{10}}\Big(\frac{p_{10}}{p_{-10}}\Big)^i+\frac{q(p_{0-1}/p_{01},1)C(i_0,j_0)}{p_{0-1}-p_{01}}\Big(\frac{p_{01}}{p_{0-1}}\Big)^j;$$

*and for $\gamma>\gamma_0$*

$$\pi_{ij}^{i_0j_0}\sim\frac{\widetilde{q}(p_{-10}/p_{10},1)\widetilde{C}(i_0,j_0)}{p_{-10}-p_{10}}\Big(\frac{p_{10}}{p_{-10}}\Big)^i.$$

*Proof.* The proof is as usual carried out via the saddle-point method in view of Lemma 3.25. Details are skipped. Note that only the pole $s_E^*$ of $\pi^{i_0j_0}(s)$ and the last term in (3.107) have a significant contribution to the asymptotics of $\pi_{ij}^{i_0j_0}$. □

*Proof of Theorem* 2.6. It follows from Lemma 3.26 and the definition of the Martin kernel. □

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