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Complete Cluster Expansion and Spectrum of the Hamiltonian for Lattice Fermion Models*

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The spectra of the Hamiltonians H for lattice models of quantum field theory have recently aroused considerable interest both in the mathematics [**11**, **3**, **5**] and physics [**1**, **4**, **7**, **10**, **12**] literature.

The purpose of this article is to transfer the results of [**8**, **9**] to the case of fermion models. As in [**11**], we will use Grassmann algebras for fermion models, although we do not know whether the use of Clifford variables (creation-annihilation operators) would lead to similar results.

Our results may be summarized briefly as follows. In Section 1, we develop the necessary apparatus for linear functionals in Grassmann algebras: conditional mathematical expectations, semi-invariants, etc. In Section 2, we construct a dynamics by means of the Osterwalder–Schräder construction (cf. [**11**]). In Section 3, we construct a small perturbation of an independent field by means of a cluster expansion. Such a vacuum cluster expansion differs only slightly from the boson case and is a standard expansion [**11**, **3**, **5**]. We use the apparatus of [**8**], by means of which it is possible to obtain uniform strong clustering estimates, and to some extent even the notation of that paper.

Proofs that are obtained by verbatim repetition of the corresponding proofs in [**8**] will not be presented here; instead, reference will be made to [**8**].

In Section 4, we prove a Markov property by means of which it is possible to reduce a physical Hilbert space to a single layer in the same way as in the purely probabilistic case.

The basic supplementary technique we will use here consists of the introduction of a new Hilbert space, first with an indefinite metric, and then with a positive metric (Sections 5 and 6). It is precisely in this Hilbert space that a cluster basis may be constructed (Section 5) in analogy with the probabilistic case. In Section 8, an operator is constructed that is

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unitarily equivalent to a tranfer matrix. In Sections 7 and 8, we construct, in explicit form, an operator that specifies this unitary equivalence. In Section 9, we prove the basic splitting theorem for the transfer matrix spectrum, in which the spectrum is decomposed into "finite particle" subspaces. A nonisotropicity condition must be imposed on the interaction. However, in Section 10, we will show how the case of an isotropic Dirac field (and its perturbations) may be reduced to the conditions of this case by means of a simple transformation.

Moreover, we prove a number of propositions, dealing with the statistical mechanics of Grassmann systems, that are of independent interest (necessary and sufficient Osterwalder–Schräder positivity condition, nonvanishing condition for a statistical sum, etc.).

1. Analysis and noncommutative probabilities in a Grassmann algebra

Invertibility. Suppose that $\mathfrak{A}(n)$ is a Grassmann algebra defined over $\mathbf{C}$ with n generators $x_1, x_2, \ldots, x_n$ and an identity element. An arbitrary element in the algebra may be written

$$f = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k}. \tag{1.1}$$

An element f is invertible (i.e., the left and right inverses exist and are equal) if and only if $f_\varnothing \neq 0$. The reciprocal element f^{-1} (if $f_\varnothing = 1$) is given by the formula

$$f^{-1} = (1 - Q)^{-1} = \sum_{n=0}^{\infty} Q^n, \tag{1.2}$$

where $Q = -(f - 1)$; in the latter sum, there are only a finite number of nonzero elements.

Differentiation. Right (left) differentiation $(\partial/\partial x_i)_r f$ $((\partial/\partial x_i)_l f)$ is defined as follows. In each term of f, x_i must be shifted to the right (left) and then deleted; but if x_i is absent from the monomial, the result is zero.

Berezin Integral [2]. We consider the Grassmann algebra $\mathfrak{A}(2n)$ with generators $x_1, x_2, \ldots, x_n, dx_n, \ldots, dx_2, dx_1$. Suppose that

$$T = \{i_1, i_2, \ldots, i_k\} \subset N = \{1, 2, \ldots, n\}, \qquad i_1 < i_2 < \cdots < i_k.$$

We let

$$x_T = x_{i_1} x_{i_2} \ldots x_{i_k}, \qquad dx_T = dx_{i_k} \ldots dx_{i_2} dx_{i_1}$$

and for a given dx_T we define the linear mapping

$$\int \cdot \, dx_T : \mathfrak{A}(n) \to \mathfrak{A}(n)$$

by the formula

$$\int x_{T'} x_T \, dx_T = x_{T'} \qquad \text{if} \quad T' \cap T = \varnothing;$$

$$\int x_{T'} \, dx_T = 0 \qquad \text{if} \quad T' \not\subset T,$$

and so on by linearity. If $T = N$, this formula defines a linear functional on $\mathfrak{A}(n)$. The integral may also be considered as a linear mapping, thus:

$$\int : \mathfrak{A}(2n) \to \mathfrak{A}(n).$$

Conditional Mathematical Expectation

Proposition 1.1. *Any linear functional* $\langle \cdot \rangle$ *defined on* $\mathfrak{A}(n)$ *in such a way that* $\langle 1 \rangle = 1$ *has the form*

$$\langle f \rangle = \frac{\int gf \, dx_N}{\int g \, dx_N} \tag{1.3}$$

for some $g \in \mathfrak{A}(n)$.

Proof. $\langle \cdot \rangle$ is defined by its values over monomials. We obtain a proof by selecting coefficients g in an obvious way.

Remark 1.1. This type of linear functional corresponds to a state (mathematical expectation) in the case of a C^*-algebra (algebra of random variables). The conditional functional defined below corresponds in many cases to the conditional mathematical expectation.

Suppose that we are given a linear functional (1.3), with $T \subset N$, such that $\int g \, dx_{N \setminus T}$ is invertible. We define the conditional mathematical expectation by

$$\langle f | T \rangle \overset{\text{def}}{=} \left(\int g \, dx_{N \setminus T} \right)^{-1} \int gf \, dx_{N \setminus T}. \tag{1.4}$$

Then it is clear that $\langle f | \varnothing \rangle = \langle f \rangle$.

Proposition 1.2. *If* $T_1 \supset T_2$ *and* $\int g \, dx_{N \setminus T_1}$ *and* $\int g \, dx_{N \setminus T_2}$ *are invertible,*

$$\langle \langle f | T_1 \rangle | T_2 \rangle = \langle f | T_2 \rangle. \tag{1.5}$$

Proof.

$$\begin{aligned} \langle \langle f | T_1 \rangle | T_2 \rangle &\overset{\text{def}}{=} \left(\int g \, dx_{N \setminus T_2} \right)^{-1} \int g \langle f | T_1 \rangle \, dx_{N \setminus T_2} \\ &= \pm \left(\int g \, dx_{N \setminus T_2} \right)^{-1} \int \left[\int g \langle f | T_1 \rangle \, dx_{N \setminus T_1} \right] dx_{T_1 \setminus T_2} \\ &= \pm \left(\int g \, dx_{N \setminus T_2} \right)^{-1} \int \left[\int gf \, dx_{N \setminus T_1} \right] dx_{T_1 \setminus T_2} = \langle f | T_2 \rangle, \end{aligned}$$

since

$$\int g\langle f \mid T_1\rangle \, dx_{N\setminus T_1} = \int g \, dx_{N\setminus T_1} \langle f \mid T_1\rangle = \int gf \, dx_{N\setminus T_1}.$$

We present one more property of the conditional mathematical expectation by analogy with the probabilistic case.

If f is a polynomial in x_i, $i \in T$, then

$$\langle fg \mid T\rangle = f\langle g \mid T\rangle,$$
$$\langle gf \mid T\rangle = \langle g \mid T\rangle f.$$

Formula for a Linear Change of Variables [**2**]. Suppose that $y_1, \ldots, y_n$ are the generators of the Grassmann algebra $\mathfrak{A}(n)$, and that $x_i = \Sigma_k a_{ik} y_k$, where $A = \|a_{ik}\|$ is a nondegenerate complex matrix. Then

$$\int f(x(y)) \, dy_N = \det A \int f(x) \, dx_N. \tag{1.6}$$

The proof follows from the equation

$$x_1 x_2 \ldots x_N = (\det A) y_1 y_2, \ldots, y_N.$$

Semi-invariants. We formulate this concept for an arbitrary associative noncommutative algebra $\mathfrak{A}$ with an identity element. Suppose that a linear functional $\langle \cdot \rangle$ is defined on $\mathfrak{A}$ with the property that $\langle 1\rangle = 1$, and let $\xi_1, \ldots, \xi_m \in \mathfrak{A}$. For any $T = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$, $i_1 < i_2 < \cdots < i_k$, we let

$$\xi_T = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}.$$

We define the semi-invariants

$$\langle \xi_T^{\prime} \rangle = \langle \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k} \rangle$$

inductively by the formula

$$\langle \xi_T \rangle = \sum \langle \xi_{T_1}^{\prime} \rangle \cdots \langle \xi_{T_q}^{\prime} \rangle, \tag{1.7}$$

where the sum is taken over all partitions $T_1 \cup T_2 \cup \cdots \cup T_q = T$. Hence by the Moebius inversion formula [**8**]

$$\langle \xi_T^{\prime} \rangle = \sum \langle \xi_{T_1} \rangle \cdots \langle \xi_{T_q} \rangle (-1)^{q-1} (q-1)!. \tag{1.8}$$

Therefore the semi-invariants are multilinear (in the general case, nonsymmetric) functions of ξ_i.

Proposition 1.3. *The formal Taylor expansion*

$$\ln\langle \exp(\lambda_1 \xi_1 + \lambda_2 \xi_2 + \cdots + \lambda_m \xi_m)\rangle = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k} \langle \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k} \rangle \tag{1.9}$$

holds for real $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Proof. As in the commutative case, we have the formal series

$$\ln\langle \exp \lambda f\rangle = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \langle f, f, \ldots, f\rangle.$$

We make the substitution

$$\lambda f = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m$$

and use the multilinearity property.

Note that the right-hand side of (1.9) actually depends on what are known as symmetrized semi-invariants, i.e.,

$$\ln\langle \exp(\lambda_1 \xi_1 + \cdots + \lambda_n \xi_n)\rangle = \sum_{m_1, m_2, \ldots, m_n} \frac{\lambda_1^{m_1} \ldots \lambda_n^{m_n}}{m_1! \ldots m_n!} \langle \xi_1^{m_1}, \ldots, \xi_n^{m_n}\rangle^s,$$

$$\text{where} \quad m_i \geqslant 0, \quad m_1 + m_2 + \cdots + m_n \geqslant 1, \tag{1.10}$$

$$\langle \xi_1^{m_1}, \ldots, \xi_n^{m_n}\rangle = \langle \underbrace{\xi_1, \ldots, \xi_1}_{m_1 \text{ times}}, \xi_2, \ldots, \underbrace{\xi_n, \ldots, \xi_n}_{m_n \text{ times}}\rangle,$$

$$\langle \xi_1, \xi_2, \ldots, \xi_{m-1}, \xi_m\rangle^s = \frac{1}{m!} \sum_{\pi} \langle \xi_{\pi(1)}, \xi_{\pi(2)}, \ldots, \xi_{\pi(m)}\rangle.$$

Here the sum is taken over all permutations of the form

$$\pi = \begin{pmatrix} 1 & \cdots & m \\ \pi(1) & \cdots & \pi(m) \end{pmatrix}.$$

Gaussian Integrals. Suppose that n is even and let $a_{ik} = -a_{ki}$. Then

$$\int \exp\left(\sum a_{ik} x_i x_k\right) dx_n \ldots dx_1 = \pm(\det\|2a_{ik}\|)^{1/2}. \tag{1.11}$$

(For the proof, see [2].) The choice of sign depends on the matrix $A = \|a_{ik}\|$. For example, we may consider the case $n = 2$ and $a_{11} = a_{22} = 0$. If $a_{12} = -a_{21} = \frac{1}{2}$, the left-hand side of (1.11) is given by

$$\int \exp(x_1 x_2)\, dx_2\, dx_1 = +1.$$

But if $a_{21} + \frac{1}{2}$, the left-hand side of (1.11) is equal to -1. In both cases, $\det(2A) = +1$. An important special case is obtained if the generators in $\mathfrak{A}(2n)$ are denoted by $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$. Then it follows from (1.11) that

$$\int \exp\left(\sum c_{ik} x_i \bar{x}_k\right) d\bar{x}_n\, dx_n \cdots d\bar{x}_1\, dx_1 = \det\|C_{ik}\|. \tag{1.12}$$

Gaussian Means and Diagrams. Suppose that the generators of $\mathfrak{A}(2n)$ are $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$. We set

$$F = x_{i_1}\bar{x}_{j_1} \cdots x_{i_k}\bar{x}_{j_k}, \tag{1.13}$$

and consider the Gaussian mean

$$\langle F\rangle = \left(\int \exp\left(\sum c_{ij}x_i\bar{x}_j\right) d\bar{x}_n\, dx_n \cdots d\bar{x}_1\, dx_1\right)^{-1}$$
$$\times \int F \exp\left(\sum c_{ij}x_i\bar{x}_j\right) d\bar{x}_n\, dx_n \cdots d\bar{x}_1\, dx_1.$$

To calculate this mean, we consider the Grassmann characteristic functional

$$\left(\int \exp\left(\sum_{i,j} c_{ij}x_i\bar{x}_j\right) d\bar{x}_n \cdots dx_1\right)^{-1}$$
$$\times \int \exp\left(\sum_{i,j} c_{ij}x_i\bar{x}_j + \sum_i \eta_i\bar{x}_i + \sum_j x_j\bar{\eta}_j\right) d\bar{x}_n \cdots dx_1$$
$$= \exp(-\eta c^{-1}\bar{\eta}). \tag{1.14}$$

Here η_i and $\bar{\eta}_j$ are Grassmann variables that, together with x_i and $\bar{x}_j$, generate the Grassmann algebra $\mathfrak{A}(4n)$. To verify (1.14), we substitute in the integral in the numerator

$$x_i \to x_i - \sum_k b_{ik}\eta_k, \qquad \bar{x}_j \to \bar{x}_j - \sum b_{jl}\bar{\eta}_l, \tag{1.15}$$

where $B = \|b_{ik}\| = C^{-1}$. This substitution does not affect the value of the integral (it does not change the coefficient of $x_1\bar{x}_1 \cdots x_n\bar{x}_n$).

Proposition 1.4.

$$\langle F\rangle = \sum_\pi b_{i_1 j_{\pi(1)}} \cdots b_{i_k j_{\pi(k)}} (-1)^{|\pi|}, \tag{1.16}$$

where the sum is taken over all permutations of the form

$$\pi = \begin{pmatrix} 1 \cdots k \\ \pi(k) \cdots \pi(k) \end{pmatrix}$$

and $|\pi|$ is the parity of π.

Proof. By (1.14),

$$\langle F\rangle = \left(\frac{\partial}{\partial\bar{\eta}_{j_k}}\right)_r \left(\frac{\partial}{\partial\eta_{i_k}}\right)_l \cdots \left(\frac{\partial}{\partial\bar{\eta}_{j_1}}\right)_r \left(\frac{\partial}{\partial\eta_{i_1}}\right)_l \exp(-\eta C^{-1}\bar{\eta}).$$

Hence (1.16) follows at once. For monomials F of a different form, we have $\langle F\rangle = 0$.

As in [8], it is useful to introduce the concept of a diagram. Suppose that the monomials $F_1, F_2, \ldots, F_p$ have the form (1.13), and that their product $F_1F_2 \cdots F_p \neq 0$. Every monomial F_i will be represented by a vertex whose set of stalks[1] corresponds to the generators that form this monomial. Then

[1] Russian, "otrostki"; called "outlets" in the translation of [8]. (Translator's note.)

in computing $\langle F_1, F_2, \ldots, F_p\rangle$ each term on the right-hand side of (1.16) corresponds to some diagram obtainable by a pairing of stalks. Here it is assumed that in the diagram every stalk corresponding to an unbarred variable is paired with a stalk corresponding to a barred variable. With every diagram we may associate a graph whose vertices are the vertices of the diagram and whose edges are paired stalks. A diagram is said to be connected if its corresponding graph is connected.

Proposition 1.5. *If every monomial F_i has the form* (1.13), *then* $\langle F_1, F_2, \ldots, F_p\rangle$ *is equal to the sum of the contributions of all connected diagrams.*

Proof. Suppose the assertion is true for all $p \leqslant m-1$. Then

$$\langle F_1, \ldots, F_m\rangle = \langle F_1 F_2 \cdots F_m\rangle - \sum_{k\neq 1} \langle F'_{T_1}\rangle \cdots \langle F'_{T_k}\rangle.$$

The first term on the right-hand side corresponds to the sum of all diagrams, while the remaining terms, by the induction hypothesis, correspond to all unconnected diagrams. Therefore, the left-hand side contains the sum of all connected diagrams.

Remark 1.2. It is interesting that Gaussian semi-invariants even of the fourth order may be nonzero, for example,

$$\langle x_1, x_2, \bar{x}_1, \bar{x}_2\rangle = \langle x_1 x_2 \bar{x}_1 \bar{x}_2\rangle - \langle x_1\bar{x}_1\rangle\langle x_2\bar{x}_2\rangle - \langle x_1\bar{x}_2\rangle\langle x_2\bar{x}_1\rangle = -2b_{11}b_{22}.$$

However, all symmetrized semi-invariants of order greater than two vanish, as follows if we compute the integral

$$\int \exp\left(\sum c_{ij}x_i\bar{x}_j + \sum \bar{\lambda}_i x_i + \sum \lambda_j \bar{x}_j\right) d\bar{x}_n \cdots dx_1 = \exp(\lambda C^{-1}\bar{\lambda}), \quad (1.18)$$

where

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \qquad \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$$

are real vectors. The calculation is similar to that in (1.14) and (1.15).

From now on we shall use another definition of semi-invariants for the Grassmann algebra. Let us suppose that a linear functional $\langle\cdot\rangle$ on $\mathfrak{A}$ satisfies the equation $\langle f\rangle = 0$ for any odd monomial $f \in \mathfrak{A}$. Let $\xi_1, \xi_2, \ldots, \xi_m$ be arbitrary monomials of the Grassmann algebra $\mathfrak{A}$. We shall use the formula

$$\langle \xi_T\rangle = \sum \sigma \times \langle \xi_{T_1}\rangle \times \cdots \times \langle \xi_{T_q}\rangle$$

instead of (1.7). The multiplier

$$\sigma = \sigma(T_1, T_2, \ldots, T_q, \xi_1, \ldots, \xi_m) = \pm 1$$

is defined by the equation

$$\xi_T = \sigma \times \xi_{T_1} \times \xi_{T_2} \times \cdots \times \xi_{T_q}.$$

Then we can define semi-invariants for any polynomial using the property of multilinearity.

Note that a number of the properties of semi-invariants are preserved in the Grassmann case, for example, the generalized expansion in connected groups (formula (1.10) of Section 1.1 [8] and Lemma 1.1 of Section 2.1 [8]).

2. Construction of the dynamics

To construct the dynamics, we repeat the Osterwalder–Schräder construction [11].

In what follows,

$$\tilde{Z}^\nu = \{t = (t^0, t^1, \ldots, t^{\nu-1})\}$$

is a ν-dimensional integer-lattice shifted by the vector $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, and $t^i = \frac{1}{2} + n$, where n is an integer. Suppose that an algebra $\mathfrak{A}_\Lambda$ (with identity element) corresponds to every finite element $\Lambda \subset \tilde{Z}^\nu$; if $\Lambda_1 \subset \Lambda_2$, then $\mathfrak{A}_{\Lambda_1} \subset \mathfrak{A}_{\Lambda_2}$. We set $\mathfrak{A} = \bigcup_\Lambda \mathfrak{A}_\Lambda$ (local algebra).

We will consider the case in which $\mathfrak{A}_\Lambda$ is a Grassmann algebra with $2s|\Lambda|$ generators

$$\psi_\alpha^1(t) = \psi_\alpha(t), \qquad \psi_\alpha^2(t) = \bar{\psi}_\alpha(t), \qquad \alpha = 1, 2, \ldots, s; \quad t \in \Lambda.$$

We let $\mathfrak{A}^+$ ($\mathfrak{A}^-$) denote the subalgebra of $\mathfrak{A}$ with generatrices $\psi_\alpha^\varepsilon(t)$ such that $t^0 > 0$ ($t^0 < 0$). Suppose that we are given an antilinear mapping $\Theta : \mathfrak{A} \to \mathfrak{A}$ with the properties

$$\Theta : \mathfrak{A}^\pm \to \mathfrak{A}^\mp, \tag{2.1}$$

$$\Theta(fg) = (\Theta g)(\Theta f). \tag{2.2}$$

Then Θ will be selected by letting

$$\vartheta(t^0, t^1, \ldots, t^{\nu-1}) = (-t^0, t^1, \ldots, t^{\nu-1})$$

denote a mapping into $\tilde{Z}^\nu$ and setting

$$\Theta\psi_\alpha^\varepsilon(t) = r(\alpha)\psi_\alpha^{3-\varepsilon}(\vartheta t), \qquad \varepsilon = 1, 2.$$

Here $r(\alpha) = \pm 1$ is arbitrary.

Remark 2.1. For the Dirac field in [11], $s = 4$ and $r(\alpha) = 1$ for $\alpha = 1, 2$, and $r(\alpha) = -1$ for $\alpha = 3, 4$. If we make the substitution

$$\psi_\alpha^1(t) \to i\psi_\alpha^2(t), \qquad \psi_\alpha^2(t) = i\psi_\alpha^1(t) \qquad (i = \sqrt{-1})$$

for $\alpha = 3, 4$, it will be clear that we may limit ourselves to the case $r(\alpha) = 1$. (The transition matrix has the form $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ with determinant 1, so that the formula for change of variables is trivial in this case.)

By property (2.3) and antilinearity, we may continue Θ to the entire algebra $\mathfrak{A}$. Now suppose that a linear functional $\langle \cdot \rangle$ is defined on $\mathfrak{A}$ with

$\langle 1\rangle = 1$. We define a bilinear form on $\mathfrak{A}^+$ by

$$(f, g) = \langle(\Theta f)\, g\rangle. \tag{2.3}$$

Suppose that the form is Hermitian and non-negative definite on $\mathfrak{A}^+$. We set

$$\mathfrak{N}^+ = \{f \in \mathfrak{A}^+ : (f, f) = 0\}.$$

Then $\mathfrak{N}^+$ is a linear space (by the Schwarz inequality). The scalar product is then defined on $\mathfrak{A}^+/\mathfrak{N}^+$. With this scalar product, the completion $\mathscr{H} = \mathscr{H}_{\text{phy}}$ to the space $\mathfrak{A}^+/\mathfrak{N}^+$ will be called a physical Hilbert space.

Suppose that in $\mathfrak{A}$ we define a representation S_t of the group Z^ν, $t \in Z^\nu$, such that $S_t : \mathfrak{A}_\Lambda \to \mathfrak{A}_{\Lambda+t}$ is an isomorphism. In our case we let

$$S_{t'}\psi_\alpha^\varepsilon(t) = \psi_\alpha^\varepsilon(t + t'),$$

which defines S_t uniquely. We let $S = S_{e_0}$, where $e_0 = (1, 0, 0, \ldots, 0)$; S carries $\mathfrak{A}^+$ into itself. Now suppose that

$$(Sf, g) = (f, Sg) \tag{2.4}$$

for all f, $g \in \mathfrak{A}^+$. Then S preserves $\mathfrak{N}^+$, since

$$0 \leqslant (Sf, Sf) = (f, S^2 f) \leqslant \|f\|\,\|S^2 f\|. \tag{2.5}$$

Therefore S is defined on $\mathscr{H}$, and if S is bounded, then S is self-adjoint; furthermore,

$$S^2 \geqslant 0. \tag{2.6}$$

If

$$|(f, S^n f)| < c(1 + n^m) \tag{2.7}$$

for some $c = c(f)$ and m, we iterate (2.5) to find that

$$\|S\| \leqslant 1. \tag{2.8}$$

Remark 2.2. Assuming that 0 is not an eigenvalue of S, we may write

$$S^2 = e^{-2H}, \qquad H \geqslant 0.$$

The dynamics are then specified by the unitary group e^{itH}. We will be concerned with the spectral properties of H (i.e., S^2).

Suppose that Λ is symmetric with respect to the hyperplane $t^0 = 0$ and let $\mathscr{K} \in \mathfrak{A}_\Lambda$. We will consider the "unnormalized" bilinear form $B(\cdot,\cdot)$ on $\mathfrak{A}_\Lambda^+ = \mathfrak{A}_\Lambda \cap \mathfrak{A}^+$:

$$\begin{aligned} B(f, g) &= \int \mathscr{K}(\Theta f) g \, d\psi_\Lambda, \\ d\psi_\Lambda &= \prod_{t\in\Lambda} \prod_{\alpha=1}^{s} \left(d\psi_\alpha^2(t)\, d\psi_\alpha^1(t)\right). \end{aligned} \tag{2.9}$$

Proposition 2.1. *The form* (2.9) *is Hermitian on* $\mathfrak{A}^+$ *if and only if* $\Theta\mathcal{K} = \mathcal{K}$.

The proof is evident.

We set

$$T = \tilde{Z}^\nu \times \{1,2,\ldots,s\} \times \{1,2\};$$

$$T^- = \{(t,\alpha,\varepsilon) \in T : t^0 < 0\};$$

$$T_{1/2} = \{(t,\alpha,\varepsilon) \in T : t^0 = \tfrac{1}{2}\}.$$

We let $\mathfrak{A}_{1/2}$ denote the subalgebra of $\mathfrak{A}^+$ with an identity element and generators $\psi_\alpha^\varepsilon(t)$, $(t,\alpha,\varepsilon) \in T_{1/2}$.

For arbitrary $A \subset T$, we let $\tilde{A}$ denote the projection of A onto $\tilde{Z}^\nu \times \{1,2,\ldots,s\}$ and $\tilde{\tilde{A}}$ its projection onto $\tilde{Z}^\nu$. It will be convenient to denote the direct product $\Lambda \times \{1,2,\ldots,s\} \times \{1,2\}$ merely by Λ. We let

$$\Lambda^- = \Lambda \cap T^-, \qquad \Lambda^+ = \Lambda \backslash \Lambda^-,$$

etc. We also write

$$d_A^0 = \max\{|t_1^0 - t_2^0 : t_1, t_2 \in \tilde{\tilde{A}}\}.$$

The following proposition was established in [**11**].

Theorem 2.1. *The set* $\mathcal{P}$ *of all even polynomials in* $\mathfrak{A}$ *of the form*

$$F = \sum_i (\Theta F_i) F_i, \qquad F_i \in \mathfrak{A}^+,$$

is closed under the operations of addition and multiplication.

Remark 2.3. There are no parity requirements in [**11**], although the proof is given only for even polynomials.

From now on we assume that $\mathcal{K}$ is even and $\mathcal{K} = \Theta\mathcal{K}$.

Theorem 2.2. *The Hermitian form* (2.9) *will be nonnegative definite if and only if* $\mathcal{K} \in \mathcal{P}$.

Proof. Let us prove the sufficiency. Suppose that $\mathcal{K} = \Sigma(\Theta G_i)G_i$ is an even polynomial in $\mathcal{P}$ and let $h \in \mathfrak{A}^+$. In this case

$$\mathcal{K}(\Theta h)h = (\Theta h)\mathcal{K}h = (\Theta h)\sum(\Theta G_i)G_i h$$
$$= \sum(\Theta(G_i h))(G_i h).$$

Using Lemma 2.2 of [**11**], we find that $B(h,h) > 0$.

Let us now prove the necessity. For every finite $M \subset T$, we introduce the notation

$$\psi_M = \prod_{(t,\alpha,\varepsilon)\in M} \psi_\alpha^\varepsilon(t),$$

where the product is taken in some order, for example, in lexicographic order. For arbitrary $h \in \mathfrak{A}_\Lambda^+$ and $\mathcal{K} \in \mathfrak{A}_\Lambda$, the expansions

$$h = \sum_{M \in \Lambda^+} h_M \psi_M, \qquad \mathcal{K} = \sum_{K \subset \Lambda^+} \sum_{M \subset \Lambda^+} b_{KM} (\Theta \psi_K) \psi_M$$

exist and are unique (here $b_{K,M}$, $h_M \in C$). Thus we have established an isomorphism between the bilinear forms $B(\cdot, \cdot)$ and the matrices $B = \{b_{K,M}\}$. Here $b_{KL} = 0$ if $|K|$ and $|L|$ have different parity, since $\mathcal{K}$ is even. We now prove that B is a Hermitian matrix and is non-negative definite. In fact,

$$\begin{aligned} 0 &\leqslant \mathcal{K}(\Theta h) h \, d\psi_\Lambda \\ &= \sum_{K,L,M,N \subset \Lambda^+} b_{KL} \bar{h}_M h_N \int \int (\Theta \psi_M)(\Theta \psi_K) \psi_L \psi_N \, d\mu_{\Lambda^-} \, d\psi_{\Lambda^+} \\ &= \sum b_{KL} \bar{h}_M h_N \Big(\int (\psi_K \psi_M) \, d\psi_{\Lambda^+} \Big) \Big(\int (\psi_L \psi_N \, d\psi_{\Lambda^+}) \Big). \end{aligned}$$

In the last sum, the first (or second) integral is nonzero if and only if $M = \Lambda^+ \setminus K$ (or $N = \Lambda^+ \setminus L$). The integrals are equal (to $+1$ or -1) if $K = L$ and $N = M = \Lambda^+ \setminus K$. Our inequality may therefore be rewritten in the form

$$0 \leqslant \sum_{K,L} b_{KL} \bar{y}_K y_M, \qquad y_M \in \mathbf{C}, \quad y_M = \pm h_M.$$

Since $\Theta \mathcal{K} = \mathcal{K}$, we have $b_{KL} = \bar{b}_{LK}$.

In $\mathfrak{A}_\Lambda^+$ we define an operator B with matrix elements $\{b_{KL}\}$ in the basis $\{\psi_K, K \subset \Lambda^+\}$; then B is a self-adjoint non-negative definite operator. There is a unique operator $A = \sqrt{B}$ which is self-adjoint and non-negative definite, and is the limit of some sequence of polynomials in B. Suppose $\{a_{KM}\}$ are the matrix elements of A in the same basis $\{\psi_K, K \subset \Lambda^+\}$. Then $a_{KM} = 0$ if $|K|$ and $|M|$ are of different parity. We set $G_K = \Sigma a_{KM} \psi_M$, then $\mathcal{K} = \Sigma(\Theta G_K) G_K$, with $K \in \mathcal{P}$. Theorem 2.3 is proved.

Let us now consider the case in which $\mathcal{K} = \exp U_\Lambda$, where

$$U_\Lambda = \sum_{A \subset \Lambda} c_A \psi_A, \qquad \bar{c}_A \in \mathbf{C}. \tag{2.10}$$

Without loss of generality, we may assume that $c_\varnothing = 0$. Let U_Λ^+ denote the sum of all the terms of U_Λ that belong to $\mathfrak{A}^+$. Then $\Theta U_\Lambda^+ \in \mathfrak{A}^-$. We set

$$U_\Lambda^0 = U_\Lambda - U_\Lambda^+ - \Theta U_\Lambda^+. \tag{2.11}$$

From the definition of (2.11), it follows that

$$U_\Lambda^0 = \sum_{A \,:\, A \cap \Lambda^+ \neq \varnothing \text{ and } A \cap \Lambda^- \neq \varnothing} c_A \psi_A,$$

where the sum is taken over all subsets $A \subset \Lambda$ that have nonempty intersections with Λ^+ and Λ^- simultaneously. We introduce the constraint

$c_A = 0$ if $d_A^0 > 1$. Then

$$U_\Lambda^0 = \sum_{K,L \in \Lambda_{1/2}} c_{KL}(\Theta\psi_K)\psi_L,$$

where $c_{KL} = c_A$ if $(\Theta\psi_K)\psi_L = \psi_A$. The sum is taken over all nonempty subsets $K, L \in \Lambda_{1/2}$. Since $\mathscr{K} \in e^{U_\Lambda} \in \mathscr{P}$, we have

$$e^{U_\Lambda^0} = \mathscr{K}(\Theta e^{-U_\Lambda^+})e^{-U_\Lambda^+} \in \mathscr{P},$$

$$e^{U_\Lambda^0} = 1 + \sum_{K,L} c'_{K,L}(\Theta\psi_K)\psi_L.$$

We now apply Theorem 2.3, assuming that $\mathscr{K} = e^{U_\Lambda^0}$, that is, we assume that $b_{KL} = c'_{K,L}$ if $K \neq \emptyset$ and $L \neq \emptyset$; $b_{\emptyset\emptyset} = 1$ and $b_{KL} = 0$ otherwise (i.e., when K or L is empty, but not both). Then

$$e^{U_\Lambda^0} = \sum_{K,L \subset \Lambda_{1/2}} b_{KL}(\Theta\psi_K)\psi_L$$

$$= \sum_{M \subset \Lambda_{1/2}} \sum_{K,L} a_{KM}a_{ML}(\Theta\psi_k)\psi_L$$

$$= \sum_M \left(\Theta \sum_K a_{MK}\psi_K\right)\left(\sum_L a_{ML}\psi_L\right).$$

By construction, $a_{\emptyset\emptyset} = 1$ and $a_{ML} = 0$ if precisely one of the sets M or L is empty, in which case

$$e^{U_\Lambda^0} = 1 + \sum_M \left(\Theta \sum_{K \neq \emptyset} a_{MK}\psi_K\right)\left(\sum_{L \neq \emptyset} a_{ML}\psi_L\right)$$

$$= 1 + \sum_M (\Theta G_M)G_M,$$

and the free constant in G_M is zero. Thus we have proved the following theorem.

Theorem 2.3. *Suppose* $\exp U_\Lambda \in \mathscr{P}$ *and let* $c_A = 0$ *if* $d_A^0 > 1$, *in which case*

$$U_\Lambda^0 = \ln\left(1 + \sum (\Theta G_M)G_M\right)$$

$$= \sum_{l=1} (-1)^{l-1}\frac{1}{l}\left(\sum_{M \subset \Lambda_{1/2}} (\Theta G_M)G_M\right)^l; \qquad (2.12)$$

furthermore, all the sums are finite, and $G_M \in \mathfrak{A}_{1/2} \cap \mathfrak{A}_\Lambda$.

3. Construction of the Gibbs field

We first define an "independent field," i.e., a linear functional $\langle \cdot \rangle_0$ on $\mathfrak{A}$, by supposing that $A \subset \Lambda$ and that F_A is supported on A, i.e., is a polynomial in the generators $\psi_\alpha^\varepsilon(t)$, $(t, \alpha, \varepsilon) \in A$. Then we may assume that

$$\langle F_A \rangle_0 = \Xi_0^{-1} \int F_A \exp V_0 \, d\psi_\Lambda,$$

$$V_0 = \sum_{(t,\alpha) \in \tilde{\Lambda}} \psi_\alpha^\varepsilon(t)\psi_\alpha^2(t), \qquad \Xi_0 = \int e^{V_0} d\psi_\Lambda = 1.$$

Note that $\langle F_A\rangle_0$ is independent of the choice of Λ, and that we need only require that $A \subset \Lambda$. The property of "independence" has the following precise meaning. If $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ are pairwise disjoint, then

$$\langle F_{A_1}F_{A_2}\cdots F_{A_n}\rangle_0 = \langle F_{A_1}\rangle_0\langle F_{A_2}\rangle_0\cdots\langle F_{A_n}\rangle_0. \tag{3.1}$$

Suppose that $\|\tilde{\tilde{A}}\| \geqslant 2$, in which case we may let

$$d_A = d_A^*.$$

Here d_A^* is the smallest sum of the lengths of edges of the connected tree constructed on $\tilde{\tilde{A}}$ (cf. [8]). Length is measured in the metric

$$|t_1 - t_2| = \sum_{i=0}^{\nu-1} |t_1^i - t_2^i|.$$

If $\|\tilde{\tilde{A}}\| < 2$, we let $d_A = 1$. We also let $d(A_1, \ldots, A_n)$ denote the smallest value of d_A, where A is such that the set $\{A, A_1, \ldots, A_n\}$ is connected.

We now define a Gibbs linear functional $\langle\cdot\rangle$ on $\mathfrak{A}_\Lambda$ by

$$\langle F_A\rangle_\Lambda = \Xi_\Lambda^{-1}\langle F_A \exp V_\Lambda\rangle_0,$$

with

$$V_\Lambda = \sum_{A\subset\Lambda} c_A\lambda^{d_A}\psi_A, \qquad \Xi_\Lambda = \langle \exp V_\Lambda\rangle_0. \tag{3.3}$$

It will always be assumed that $|\lambda|$ is sufficiently small and that $|c_A| < c$. Then $\Xi_\Lambda > 0$. The set of numbers c_A will be assumed to be translation-invariant.

If there exists a linear functional $\langle\cdot\rangle$ on $\mathfrak{A}$ which is a weak limit point of $\langle\cdot\rangle_\Lambda$, then $\langle\cdot\rangle$ will be called the limiting Gibbs state for the formal operation

$$\sum\psi_\alpha^1(t)\psi_\alpha^2(t) + \sum\tilde{c}_A\psi_A, \qquad \tilde{c}_A = c_A\lambda^{d_A}. \tag{3.4}$$

We now present the concept of a cluster expansion in a case more general than that treated in [8]. Suppose we are given the direction (increasing) of a family of finite sets $\Lambda\nearrow\tilde{Z}^\nu$ and a sequence of linear functionals defined on $\mathfrak{A}_\Lambda$.

Definition. Suppose that for every $F_A \in \mathfrak{A}_A$ we have $|A| < \infty$ and that for any $\Lambda \supset A$ in our family we have

$$\langle F_A\rangle_\Lambda = \sum_R b_R^{(\Lambda)}, \tag{3.5}$$

where the sum is taken over all finite $R \subset \Lambda$ and the numbers $b_R^{(\Lambda)} = b_R^{(\Lambda)}(F_A)$ possess the following properties:

C1. $b_R^{(\Lambda)}$ converges to certain numbers b_R for all R, with $\Lambda\nearrow T$:

C2. $b_R^{(\Lambda)}$ are bounded uniformly with respect to Λ by the numbers $\hat{b}_R$, and the series $\Sigma\hat{b}_R$ converges.

Then

$$\langle F_A\rangle = \lim_{\Lambda\nearrow T}\langle F_A\rangle_\Lambda = \sum b_R(F_A)$$

exists and defines a linear functional on $\mathfrak{A}$. We say that there exists a (vacuum) cluster expansion for this linear functional.

We will also require the concept of an exponentially regular cluster expansion, which may be translated word for word from Section 1.2 [8] to our somewhat more general case, and so will not be repeated here.

The existence of a cluster expansion in the domain of interest is already quite well known. It may be obtained in a variety of ways: for example, if all the ψ_A are even, then any of the methods given in [11, 3, 5] may be applied.

Theorem 3.1. *For sufficiently small $\lambda_0 > 0$, a cluster expansion in the domain of interest ($|\lambda| < \lambda_0$, $|c_A| < c$) exists and the $\langle F_B \rangle$ are analytic functions of λ and every c_A in this domain.*

To prove this assertion for odd ψ_A, we may use such methods as expanding $\langle F_B \rangle$ in a series in semi-invariants and then proving the convergence of these series, using the bounds found for semi-invariants in Section 2.1 [8]. However, we need only deal with the case in which all the ψ_A are even, and so we may use the method given in Section 1.4 [8]. As in [8], we let

$$k_A = \exp(c_A \psi_A) - 1 = c_A \psi_A, \qquad k_\Gamma = \prod_{A \subset \Gamma} k_A .$$

It is easily seen that

$$|\langle k_\Gamma \rangle_0| < |c\lambda|^{|\Gamma|}, \tag{3.6}$$

where $c = \max|c_A|$. We shall use the notation of [8]. Repeating the computations of Section 1.4 [8] verbatim, we find that

$$\langle F_B \rangle = \sum_\gamma a_\gamma ,$$

$$a_\gamma = \langle F_B k_{\Gamma_1(\gamma)} \rangle_0 \cdots \langle - k_{\Gamma_n(\gamma)} \rangle_0 . \tag{3.7}$$

Theorem 3.2. *The cluster expansion obtained above is exponentially regular, and we have the uniform strong clustering estimates*

$$|\langle F_{B_1}, F_{B_2}, \ldots, F_{B_n} \rangle| \leqslant c^{\Sigma|B_i|} |c\lambda|^{d(B_1, B_2, \ldots, B_n)} \prod_{i=1}^{n} u_i , \tag{3.8}$$

where u_i is the number of B_j ($j = 1, 2, \ldots, n$) for which $B_i \cap B_j \neq \emptyset$.

Proof. The existence of exponential regularity is proved exactly as done at the end of Section 1.4 [8]. The existence of uniform strong clustering estimates is also proved exactly as in [8], i.e., the inequalities (3.6) in Chapter II of [8]. From these inequalities, (3.8) follows if we can prove that

$$|\langle F_B \rangle| \leqslant C^{|B|} \tag{3.9}$$

for every monomial F_B with coefficient 1 and some constant C which is independent of B. But the bound in (3.9) follows from the cluster expansion (3.7) and Lemma 3.4 of Section 1.3 [**8**]. Substituting (3.9) for c_ψ in (3.6) of Section 2.3 [**8**] yields the desired result.

4. Markov property and reduction of physical Hilbert space

In what follows we will assume that the conditions of the preceding section hold, along with condition (2.12). On $\mathfrak{A}$ we define the bilinear form

$$(f, g) = \langle (\Theta f)\, g \rangle, \tag{4.1}$$

where $\langle \cdot \rangle$ is the limiting Gibbs functional constructed in Section 3.

Lemma 4.1. *For all* $f \in \mathfrak{A}^+$,

$$(f, f) \geqslant 0,$$

and the inequalities (2.4) *and the bound* (2.7) *hold.*

Proof. The non-negativity follows from Theorem 2.2. In passing to the limit, the volumes Λ are chosen to be invariant with respect to ϑ. In place of (2.7), we have the stronger bound

$$|(f, S^n f)| \leqslant |\langle (\Theta f) S^n f \rangle| \leqslant \text{const},$$

which follows from (3.9).

Lemma 4.2 (*Markov Property*). *Let*

$$\int e^{U_\Lambda}\, d\psi_{\Lambda \backslash \Lambda_{1/2}}$$

be invertible and $f \in \mathfrak{A}^+$. *Then*

$$\langle f \,|\, \Lambda_{1/2} \cup \Lambda^- \rangle_\Lambda = \langle f \,|\, \Lambda_{1/2} \rangle_\Lambda . \tag{4.2}$$

Proof. We use the representation (2.11):

$$U_\Lambda = U_\Lambda^+ + U_\Lambda^0 + \Theta U_\Lambda^+ .$$

Then

$$\begin{aligned}
\langle f \,|\, \Lambda_{1/2} \rangle_\Lambda &= \Big(\int \exp U_\Lambda\, d\psi_{\Lambda \backslash \Lambda_{1/2}} \Big)^{-1} \int f \exp U_\Lambda\, d\psi_{\Lambda \backslash \Lambda_{1/2}} \\
&= \Big(\int \exp(U_\Lambda^0 + \Theta U_\Lambda^+)\, d\psi_{\Lambda^-} \int \exp U_\Lambda^+\, d\psi_{\Lambda^+ \backslash \Lambda_{1/2}} \Big)^{-1} \\
&\quad \times \int \exp(U_\Lambda^0 + \Theta U_\Lambda^+)\, d\psi_{\Lambda^-} \int f \exp U_\Lambda^+\, d\psi_{\Lambda^+ \backslash \Lambda_{1/2}} \\
&= \Big(\int \exp U_\Lambda^+\, d\psi_{\Lambda^+ \backslash \Lambda_{1/2}} \Big)^{-1} \int f \exp U_\Lambda^+\, d\psi_{\Lambda^+ \backslash \Lambda_{1/2}} \\
&= \Big(\int \exp U_\Lambda\, d\psi_{\Lambda^+ \backslash \Lambda_{1/2}} \Big)^{-1} \int f \exp U_\Lambda\, d\psi_{\Lambda^+ \backslash \Lambda_{1/2}} \\
&= \langle f \,|\, \Lambda_{1/2} \cup \Lambda^- \rangle_\Lambda .
\end{aligned}$$

Lemma 4.3. *If $g \in \mathfrak{A}_\Lambda^+$ and $f \in \mathfrak{A}_\Lambda^+$, then under the hypotheses of Lemma 4.2 we have (with all averages are taken over Λ):*

$$\begin{aligned}(g, f) &= (g, \langle f \mid \Lambda_{1/2}\rangle) \\ &= (\langle g \mid \Lambda_{1/2}\rangle, \langle f \mid \Lambda_{1/2}\rangle).\end{aligned} \tag{4.4}$$

Proof.

$$\begin{aligned}(g, f) &= \langle(\Theta g)f\rangle = \langle\langle(\Theta g)f \mid \Lambda_{1/2} \cup \Lambda^-\rangle\rangle \\ &= \langle(\Theta g)\langle f \mid \Lambda_{1/2} \cup \Lambda^-\rangle\rangle = \langle(\Theta f)\langle f \mid \Lambda_{1/2}\rangle\rangle \\ &= (g, \langle f \mid \Lambda_{1/2}\rangle\rangle = \overline{\langle(\Theta(\langle f \mid \Lambda_{1/2}\rangle))g\rangle} \\ &= \overline{\langle\langle(\Theta\langle f \mid \Lambda_{1/2}\rangle)g \mid \Lambda_{1/2} \cup \Lambda^-\rangle\rangle} = \overline{\langle(\Theta\langle f \mid \Lambda_{1/2}\rangle)\langle g \mid \Lambda_{1/2} \cup \Lambda^-\rangle\rangle} \\ &= \overline{(\Theta\langle f \mid \Lambda_{1/2}\rangle)\langle g \mid \Lambda_{1/2}\rangle\rangle} \\ &= \langle(\Theta\langle g \mid \Lambda_{1/2}\rangle)\langle f \mid \Lambda_{1/2}\rangle\rangle = (\langle g \mid \Lambda_{1/2}\rangle, \langle f \mid \Lambda_{1/2}\rangle).\end{aligned}$$

Let $\mathfrak{A}_{1/2}$ denote a subalgebra of $\mathfrak{A}$ with generators $\psi_\alpha^\varepsilon(t)$, $t^0 = \frac{1}{2}$.

Lemma 4.4. *If $f \in \mathfrak{A}_\Lambda^+$, then*

$$\langle f \mid \Lambda_{1/2}\rangle_\Lambda \in \mathfrak{A}_{1/2}.$$

Proof. This follows from (4.3):

$$\langle f \mid \Lambda_{1/2}\rangle_\Lambda = \left(\int e^{U_\Lambda^+}\, d\psi_{\Lambda^+\setminus\Lambda_{1/2}}\right)^{-1} \int f e^{U_\Lambda^+}\, d\psi_{\Lambda^+\setminus\Lambda_{1/2}}.$$

We now turn to the conditions of Section 3. Note that the conditions of Lemma 4.2 hold. This is readily verified by means of the cluster expansion for

$$\int e^{U_\Lambda}\, d\psi_{\Lambda\setminus\Lambda_{1/2}}$$

(as in Section 3).

We introduce the norm $\|\cdot\|_c$ in $\mathfrak{A}$ by

$$\left\|\sum c_A \psi_A\right\|_C = \sum |c_A| C^{|A|},$$

where the constant C is defined by (3.9), and then

$$|\langle f\rangle| \leqslant \|f\|_c.$$

We let $\overline{\mathfrak{A}}$, $\overline{\mathfrak{A}}_{1/2}$, and $\overline{\mathfrak{A}}^+$ denote the closures of the corresponding algebras with respect to the norm $\|\cdot\|_c$.

Passing to the limit as $\Lambda \nearrow T$, we obtain analogs of Lemmas 4.2–4.4 for the case of infinite volume. For example, the analog of Lemma 4.2 is as follows:

The limits

$$\lim_{\Lambda\nearrow T} \langle f \mid \Lambda_{1/2}\rangle_\Lambda \quad \text{and} \quad \lim_{\Lambda\nearrow T} \langle f \mid \Lambda_{1/2} \cup \Lambda^-\rangle_\Lambda$$

exist for all $f \in \overline{\mathfrak{A}}^+$, and are equal. Note that even if $f \in \mathfrak{A}^+$, we can say only that these limits belong to $\overline{\mathfrak{A}}_{1/2}$.

Let φ denote the mapping of $\mathfrak{A}^+$ into $\overline{\mathfrak{A}}_{1/2}$ given by the formula

$$\varphi(f) = \langle f | T_{1/2} \rangle = \lim_{\Lambda \nearrow T} \langle f | \Lambda_{1/2} \rangle_\Lambda .$$

Then

$$\varphi(\overline{\mathfrak{A}}^+) = \overline{\mathfrak{A}}_{1/2}, \qquad \varphi(\overline{\mathfrak{N}}^+) \subset \overline{\mathfrak{N}}^+ \cap \overline{\mathfrak{A}}_{1/2}. \tag{4.6}$$

Moreover, it follows from (4.4) that if $f \in \overline{\mathfrak{A}}^+$, we have

$$(f - \varphi(f), f - \varphi(f)) = 0. \tag{4.7}$$

In other words, $f - \varphi(f) \in \overline{\mathfrak{N}^+}$. Therefore we have proved the following theorem.

Theorem 4.1. *The mapping φ induces an isomorphism from $\overline{\mathfrak{A}^+}/\overline{\mathfrak{N}^+}$ onto $\mathfrak{A}_{1/2}/(\mathfrak{N}^+ \cap \mathfrak{A}_{1/2})$ that preserves scalar products.*

Therefore the physical space $\mathscr{H}$ may be identified with the closure $\overline{\mathfrak{A}_{1/2}/(\mathfrak{N}^+ \cap \mathfrak{A}_{1/2})}$, a result which is very useful in studying the evolutionary spectrum.

5. Construction of a cluster basis

Unfortunately, it is not possible to construct a cluster basis directly in $\mathscr{H}_{\text{phy}}$ as in the real case. We introduce a different bilinear form in $\overline{\mathfrak{A}^+}$:

$$(f, g)_i = \langle (S\Theta f) g \rangle .$$

It is, generally speaking, indefinite. We can see that this form is Hermitian. In fact,

$$\begin{aligned}
(f, g)_i &= \langle (S\Theta f) g \rangle \\
&= \overline{\langle \Theta((S\Theta f) g) \rangle} = \overline{\langle (\Theta g)(\Theta S \Theta f) \rangle} \\
&= \overline{\langle (S\Theta g)(S\Theta S\Theta f) \rangle} = \overline{\langle (S\Theta g)(\Theta^2 f) \rangle} \\
&= \overline{\langle (S\Theta g) f \rangle} = \overline{(g, f)_i} \,.
\end{aligned}$$

Here we have used translation-invariance and the formula $S\Theta S = \Theta$. Note also that

$$(Sf, g)_i = (f, g) = \langle (S\Theta f)(Sg) \rangle = (f, Sg)_i ,$$

i.e., S is self-adjoint. We write $x_1 < x_2$ if either $t_1 < t_2$ in the lexicographic sense, or $t_1 = t_2$ and $\alpha_1 < \alpha_2$, in which case we assume that $x_j = (t_j, \alpha_j)$. Suppose that $x = (t, \alpha)$ and let $t \in T_{1/2}$. We set

$$T_x = \{ (y, \varepsilon) \in T_{1/2} : y < x \} .$$

For arbitrary $g \in \overline{\mathfrak{A}}_{1/2}$, we set

$$\langle g | T_x \rangle = \lim_{\Lambda \nearrow T} \langle g | T_x \cap \Lambda \rangle_\Lambda . \tag{5.3}$$

Lemma 5.1. *The limit on the right-hand side of* (5.3) *exists in the sense of convergence in a quasilocal algebra.*

Proof. As in (3.7), we obtain a cluster expansion for the conditional mathematical expectation.

We set

$$U_{\Lambda,x} = U_\Lambda - \sum_{A \subset T_x \cap \Lambda} \tilde{c}_A \psi_A$$

and let $\langle \cdot \rangle_{0,x}$ denote the average with respect to the independent field in $\Lambda \setminus T_x$. Then

$$\begin{aligned}\langle g \mid T_x \cap \Lambda \rangle_\Lambda &= (\langle \exp U_\Lambda \rangle_{0,x})^{-1} \langle g \exp U_\Lambda \rangle_{0,x} \\ &= (\langle \exp U_{\Lambda,x} \rangle_{0,x})^{-1} \langle g \exp U_{\Lambda,x} \rangle_{0,x} \end{aligned} \tag{5.4}$$

As in [8], we have the cluster expansion (in the notation of [8]):

$$\begin{aligned}\langle \eta \mid T_x \rangle &= \sum a_\gamma , \\ a_\gamma &= \langle \eta k_{\Gamma_1(\gamma)} \rangle_{0,x} \cdots \langle - k_{\Gamma_q(\gamma)} \rangle_{0,x} .\end{aligned} \tag{5.5}$$

Notice that all the $\langle - k_{\Gamma_\kappa(\gamma)} \rangle_{0,x}$ are even polynomials in the generators of T_x. Therefore the order of the factors in 5.5 is immaterial. The term $\langle k_{\Gamma_\kappa(\gamma)} \rangle_{0,x}$ has even parity, since polynomials which are odd with respect to the variables in T_x will also be odd with respect to the variables in $\Lambda \setminus T_x$ and will yield zero after the application of $\langle \cdot \rangle_{0,x}$.

For finite $I \subset T_x$ (I may be empty), we set

$$\eta_I = \sum_{\gamma \,:\, \operatorname{supp} \gamma \cap T_x = I} a_\gamma . \tag{5.6}$$

Lemma 5.2.

$$\langle g \mid T_x \rangle = \sum_{I \subset T_x} \eta_I . \tag{5.7}$$

Here the parity of η_I coincides with the parity of g (if g is a monomial), and

$$\|\eta_I\|_c < (c\lambda)^{d_I}. \tag{5.8}$$

Furthermore, as in Section 3.1 of [8], we look for a "conditionally orthonormal" basis with respect to the scalar product introduced above. We let

$$\begin{aligned} g_x^0 &= \psi_\alpha^1(t)\psi_\alpha^2(t) - 1, \\ g_x^\varepsilon &= \psi_\alpha^\varepsilon(t), \qquad x = (t, \alpha), \quad \varepsilon = 1,2. \end{aligned} \tag{5.9}$$

Note that $\{1, g_x^0, g_x^1, g_x^2\}$ is an orthonormal basis in $\mathfrak{A}_{\{x\}}$ with scalar product $(f, g)_0 = \langle (S\Theta g)f\rangle_0$. Here

$$(g_x^\varepsilon, g_{x'}^{\varepsilon'})_i = \gamma_{\varepsilon\varepsilon'} = \begin{cases} 1, & \varepsilon = \varepsilon' = 2; \\ -1, & \varepsilon = \varepsilon' = 1,0; \\ 0, & \varepsilon \neq \varepsilon'. \end{cases} \tag{5.10}$$

We now introduce the quantities $f_x^\varepsilon \in \overline{\mathfrak{A}_{1/2}}$ with the properties

$$\begin{gathered} S\Theta f_x^\varepsilon = f_x^{3-\varepsilon}, \qquad \varepsilon = 1,2; \\ S\Theta f_x^0 = f_x^0, \qquad \langle f_x^\varepsilon \mid T_x\rangle = 0, \\ \langle (S\Theta f_x^\varepsilon) f_{x'}^{\varepsilon'} \mid T_x\rangle = \delta_{xx'}\gamma_{\varepsilon\varepsilon'}, \qquad x \geqslant x'. \end{gathered} \tag{5.11}$$

Our method of computation is the same as in [**8**]. That is, we look for f_x^ε as a perturbation of g_x^ε. We immediately have the formula

$$\begin{aligned} f_x^0 &= \frac{g_x^0 - \langle g_x^0 \mid T_x\rangle}{1 + \langle g_x^0 \mid T_x\rangle}, \\ f_x^\varepsilon &= \frac{g_x^\varepsilon - \langle g_x^\varepsilon \mid T_x\rangle (1 + g_x^0)(1 + \langle g_x^0 \mid T_x\rangle)^{-1}}{(1 + \langle g_x^0 \mid T_x\rangle)^{1/2}}. \end{aligned} \tag{5.12}$$

The square root of $1 + \langle g_x^0 \mid T_x\rangle$ and its inverse element exist, since

$$\|\langle g_x^0 \mid T_x\rangle\|_C < c\lambda.$$

All the equations in (5.11) may be verified directly.

For arbitrary finite $I \subset T_{1/2}$, we introduce the following function defined on $\tilde{I}$:

$$\varepsilon_I(x) = \begin{cases} 0 & \text{if } \; x \times \{1,2\} \subset I; \\ 1 & \text{if } \; (x,1) \in I, \;\; (x,2) \notin I; \\ 2 & \text{if } \; (x,2) \in I, \;\; (x,1) \notin I. \end{cases}$$

We set

$$f_I = \prod_{x \in \tilde{I}} f_x^\varepsilon, \qquad f_\varnothing = 1,$$

where the product is taken in lexicographic order over $x \in \tilde{I}$, $\varepsilon = \varepsilon_I(x)$.

Lemma 5.3. *The system* $\{f_I\}$ *is orthonormal, that is,*

$$(f_I, f_{I'})_i = \gamma_{II'} = \begin{cases} 0, & I \neq I', \\ \pm 1, & I = I', \end{cases}$$

where the plus (minus) sign corresponds to evenness (oddness) of the number of times $\bar{\varepsilon}_I(x)$ *is equal to* 0 *or* 1.

Proof.

$$(f_I, f_{I'}) = \langle (S\Theta f_I) f_{I'} \rangle$$
$$= \langle S\Theta f_{x_n}^{\varepsilon_n}) \cdots (S\Theta f_{x_1}^{\varepsilon_1}) f_{x_1'}^{\varepsilon_1'} \cdots f_{x_m'}^{\varepsilon_m'} \rangle. \tag{5.13}$$

We will assume that $x_1 = (t_1, \alpha_1)$ is the maximal x_i and analogously for $x_1' = (t_1', \alpha_1')$. There are three cases to consider. In the first case, $x_1 \neq x_1'$. For example, suppose that $x_1 > x_1'$. Then writing $\langle \cdot \rangle$ in the right-hand side of (5.13) in the form $\langle\langle \cdot \mid T_x \rangle\rangle_1$, we obtain zero (because of the second row of (5.11)). Similarly we obtain zero if $x_1 = x_1'$, but $\varepsilon_1 \neq \varepsilon_1'$. But if $x_1 = x_1' = x$ and $\varepsilon_1 = \varepsilon_1'$, the right-hand side of (5.13) becomes

$$\langle (S\Theta f_{x_n}^{\varepsilon_n}) \cdots (S\Theta f_{x_2}^{\varepsilon_2}) f_{x_2'}^{\varepsilon_2'} \cdots f_{x_m'}^{\varepsilon_m'} \langle (S f_{x_1}^{\varepsilon_1}) f_{x_1}^{\varepsilon_1'} \mid T_{x_1} \rangle\rangle$$
$$= \langle (S\Theta f_{x_n}^{\varepsilon_n}) \cdots (S\Theta f_{x_2}^{\varepsilon_2}) f_{x_2'}^{\varepsilon_2'} \cdots f_{x_m'}^{\varepsilon_m'} \rangle \gamma_{\varepsilon_1 \varepsilon_1}.$$

The proof of Lemma 5.3 is completed by induction.

6. Construction of a new Hilbert space and the operator $\mathscr{F}$

We let

$$\mathfrak{N}_i = \{ f \in \overline{\mathfrak{A}^+} : (f, g)_i = 0 \}$$

for all $g \in \overline{\mathfrak{A}^+}$, and prove an analog of Lemma 4.3.

Lemma 6.1. *If $f \in \overline{\mathfrak{A}^+}$, we have*

$$f - \langle f \mid T_{1/2} \rangle \in \mathfrak{N}_i. \tag{6.1}$$

Proof.

$$(g, \langle f \mid T_{1/2} \rangle)_i = \langle (S\Theta g) \langle f \mid T_{1/2} \rangle \rangle = \langle (S\Theta g) \langle f \mid T_{1/2} \cup T^- \rangle \rangle$$
$$= \langle \langle (S\Theta g) f \mid T_{1/2} \cup T^- \rangle \rangle = \langle (S\Theta g) f \rangle$$
$$= (g, f)_i.$$

Let us now consider the linear space $L = \overline{\mathfrak{A}^+} / \mathfrak{N}_i$.

Lemma 6.2. *Every element $f \in L$ may be represented in the form*

$$f = \sum_{I \subset T_{1/2}} c_I f_I, \qquad \text{where} \quad \sum_{I \subset T} |c_I| \cdot C^{|I|} < \infty. \tag{6.2}$$

Here f_I is understood to mean the image of f_I defined in Section 5 by the mapping $\overline{\mathfrak{A}^+} \to L$.

Proof. We need only to prove that an arbitrary element $f \in \mathfrak{A}_{1/2}$ may be represented in the form (6.2); the desired result will then following from the preceding lemma. But this proof is entirely analogous to the proof of Lemma 2.3 in **[8]**.

Let f be written in the form (6.2). We set

$$f_+ = \sum_{I : (f_I, f_I)_i > 0} c_I f_I, \qquad f_- = f - f_+, \tag{6.3}$$

and introduce a new scalar product in L. For f and g written in the form (6.3), we let

$$(f, g)_{\text{new}} = (f_+, g_+)_i - (f_-, g_-)_i.$$

The completion of L in this new norm is denoted by $\mathscr{H}_{\text{new}}$. If we introduce an operator δ in $\mathscr{H}_{\text{new}}$ according to the formula

$$\delta f = f_+ - f_-,$$

it is clear that

$$(f, g)_{\text{new}} = (f, \delta g)_i, \qquad (f, g)_i = (f, \delta g)_{\text{new}}.$$

Lemma 6.3. *The operator* $S : \overline{\mathfrak{A}^+} \to \overline{\mathfrak{A}^+}$ *preserves* $\mathfrak{N}_i$.

Proof. Let $f \in \mathfrak{N}_i$ and $g \in \overline{\mathfrak{A}^+}$. Then

$$(Sf, g)_i = \langle (S\Theta Sf)\, g\rangle = \langle (\Theta f)\, g\rangle = \langle (S\Theta f) Sg\rangle = (f, Sg)_i = 0.$$

By virtue of this lemma, S is defined on L. We define the operator $\mathscr{T} = \delta S$ on L, noting that δ carries L into itself.

7. Cluster properties of $\mathscr{T}$

We introduce the concept of a regular partitioning of a finite subset in T. The collection $\alpha = (A_1, A_2, \ldots, A_k)$ is called a regular partitioning of $A \subset T$ if (a) $A_1 \cup A_2 \cup \cdots \cup A_k = A$, and (b) $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k$ are pairwise disjoint.

Remark 7.1. If α is a regular partitioning of A, we have (see Section 3)

$$\langle \psi_A \rangle_0 = \langle \psi_{A_1} \rangle_0 \langle \psi_{A_2} \rangle_0 \cdots \langle \psi_{A_k} \rangle_0.$$

Let A be given and let $(A_1, A_2, \ldots, A_n)$ be the finest regular partitioning of A. Then either $A_i = (x_i, \varepsilon_i)$ or $A_i = \{(x_i, 1), (x_i, 2)\}$. In the second case, we provisionally set $\varepsilon_i = 0$ and, with this notation, set

$$\omega(A) = \langle f_{x_1}^{\varepsilon_1}, \ldots, f_{x_n}^{\varepsilon_n} \rangle.$$

Later it will be useful to define Θ on T in accordance with its earlier definition:

$$\Theta(t, \alpha, \varepsilon) = (\vartheta t, \alpha, 3 - \varepsilon).$$

Lemma 7.1. *The operator* $\mathscr{T}$ *is a (bounded) non-negative cluster operator on* $\mathscr{H}_{\text{new}}$ *in the basis* $\{f_I\}$.

Proof. Let us first prove that $\mathcal{F}$ has cluster estimates in the basis $\{f_I\}$:

$$(f_I, \mathcal{F} f_{I'})_{\text{new}} = (f_I, Sf_{I'})_i = (f_I, f_{I'})$$
$$= \langle (\Theta f_I) f_{I'} \rangle = \sum \omega(Y_1) \cdots \omega(Y_k).$$

The fact that $\omega(Y) = 0$ for $Y \subset I$ or $Y \subset \Theta I$ is proved precisely as in **[8]**. The cluster estimates

$$|\omega(Y)| < (c\lambda)^{d_Y}$$

follow from the general theorem in **[8]** on cluster estimates in the situation in which there exists an exponentially regular cluster expansion.

Observe also that $\mathcal{F}(1) = 1$. From the cluster property, it follows that $\mathcal{F}$ is bounded, and therefore has a unique continuation to $\mathcal{H}_{\text{new}}$. Let us prove that $\mathcal{F} = \mathcal{F}^*$ and $\mathcal{F} \geqslant 0$. In fact, the first assertion follows from the sequence of equations

$$(f, \mathcal{F} g)_{\text{new}} = (f, Sg)_i = (Sf, g)_i$$
$$= (\delta \mathcal{F} f, g)_i = (\mathcal{F} f, g)_{\text{new}}$$

and the second from the sequence

$$(f, \mathcal{F} f)_{\text{new}} = (f, Sf)_i = \langle (\Theta f) f \rangle = (f, f) \geqslant 0.$$

From now on we will consider a system with interaction of the form (3.4), and will require a somewhat stronger condition on the coefficients $\tilde{c}_A$:

$$\tilde{c}_A = c_A \lambda^{\tilde{d}_A}. \tag{7.1}$$

To determine the exponents $\tilde{d}_A$, we introduce a metric $\rho_A(\cdot, \cdot)$ on T for every finite subset $A \subset T$:

$$\rho_A(x_1, x_2) = \begin{cases} 1 & \text{if } \ t_1 = t_2 \pm e_0 \ \text{ and} \\ & (\{x_1, x_2\} \times \{1, 2\}) \cap A = \{(x_1, \varepsilon), (x_2, 3 - \varepsilon)\}, \\ \max\{2 + |t_1^0 - t_2^0|, |t_1 - t_2|\} & \text{otherwise.} \end{cases}$$

Let $\tilde{d}_A$ be the length of the minimal connected graph in $\tilde{T}$ whose vertex set contains $\tilde{A}$; the length of the graph is measured in the metric $\rho_A(x_1, x_2)$. If $|\tilde{A}| = 1$, we will have $\tilde{d}_A = 2$.

Let $c_{(\alpha, \varepsilon)} = c_A$ if

$$A = \{(t, \alpha, \varepsilon), (t + e_0, \alpha, 3 - \varepsilon)\}.$$

Here $\tilde{d}_A = 1$. By translation-invariance, $c_{(\alpha,\varepsilon)}$ is independent of $t \in \tilde{Z}^\nu$. From the condition $U_\Lambda = \Theta U_\Lambda$ and Theorem 2.3, it follows that all the $c_{(\alpha,\varepsilon)}\lambda$ are real and non-negative. We will limit ourselves to the case in which

$$c_{(\alpha,\varepsilon)} > 0 \tag{7.2}$$

for all $2s$ pairs (α, ε), and λ is sufficiently small and positive.

We now define the part $\mathscr{B}$ of $\mathscr{F}$ by means of its matrix elements:

$$\mathscr{B}_{II'} = (f_I, \mathscr{B} f_{I'}) = \begin{cases} 0 & \text{if } I \neq I', \\ \prod_{i=1}^{k} \omega((\vartheta J_i) \cup J_i), & I = I', \end{cases}$$

where I and I' are finite subsets of T, and the product is taken over all elements of the finest regular partitioning $(J_1, J_2, \ldots, J_k)$ of the set $I = I'$.

Lemma 7.2. *$\mathscr{B}_{II'}$ satisfes the inequalities*

$$(c_1\lambda)^{|I|} \leqslant \mathscr{B}_{II'} \leqslant (c_2\lambda)^{|I|}. \tag{7.4}$$

Proof. Let us find coefficients c_1 and c_2 such that

$$0 < 2c_1 < |c_{(\alpha,\varepsilon)}| < \tfrac{1}{2}c_2$$

for all pairs (α, ε). (There are $2s$ such pairs.)

If $|J| = 1$, $J = \{(x, \varepsilon)\}$, we will have

$$\omega((\vartheta J) \cup J) = (f_x^\varepsilon, f_x^\varepsilon) = \lambda c_{(\alpha,\varepsilon)} + O(\lambda^2),$$

and if $|J| = 2$, $J = \{(x,1),(x,2)\}$, we will have

$$\omega((\vartheta J) \cup J) = (f_x^0, f_x^0) = \lambda^2 c_{(\alpha,1)} c_{(\alpha,2)} + O(\lambda^3).$$

In both cases, for sufficiently small λ, we will have

$$(c_1\lambda)^{|J|} \leqslant \omega((\vartheta J) \cup J) \leqslant (c_2\lambda)^{|J|}.$$

The required bound follows from the definition of $\mathscr{B}_{II'}$.

Lemma 7.3. *Let us write*

$$\hat{a}_{II'} = \mathscr{F}_{II'} - \mathscr{B}_{II'}.$$

Then

$$\sum_{I' : |I'| = m} |\hat{a}_{II'}| \leqslant \begin{cases} (c\lambda)^{m+1} & \textit{if } |I| = m, \\ (c\lambda)^{\max\{|I|, m\}} & \textit{if } |I| \neq m. \end{cases}$$

Proof. We have

$$\mathscr{F}_{IJ} = \sum_{(J_1, \ldots, J_s)} \prod_{i=1}^{s} \omega(J_i),$$

where the sum is taken over all regular partitionings $(J_1, J_2, \ldots, J_s)$ of the set $(\Theta I) \cup J$. In this case, if $J_i \in \Theta I$ or $J_i \subset J$, then $\omega(J_i) = 0$. In the general case,

$$|\omega(A_0)| < (c\lambda)^{\min\{\tilde{d}_{A_1} + \tilde{d}_{A_2} + \cdots \tilde{d}_{A_m}\}}.$$

The minimum is taken over all collections $\{A_1, A_2, \ldots, A_m\}$ of subsets of

T such that:

(a) $\{\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_m\}$ is a connected graph in $\tilde{T}$;

(b)

$$\sum_{i=0}^{m} |(x,1) \cap A_i| = \sum_{i=0}^{m} |(x,2) \cap A_i|$$

for every $x \in \tilde{T}$;

(c)

$$\bigcup_{i=1}^{m} \tilde{A}_i \supset \tilde{A}_0 .$$

The rest of the proof is similar to the proof of Lemma 3.1 in [**9**]. The validity of these bounds may be verified by means of the same reasoning as in [**8**].

8. Equivalence of $\mathscr{H}_{\text{phy}}$ and $\mathscr{H}_{\text{new}}$

Suppose that the hypotheses of Section 7 are satisfied.

Lemma 8.1. $\mathfrak{N}_i \subset \overline{\mathfrak{N}^+}$.

Proof. If $f \in \mathfrak{N}_i$, then $(f, Sf)_i = 0$. Note that

$$(f, Sf)_i = (f, f).$$

Since $\mathfrak{N}_i$ and $\overline{\mathfrak{N}^+}$ are linear spaces,

$$\frac{\overline{\mathfrak{A}^+}}{\overline{\mathfrak{N}^+}} = \frac{(\overline{\mathfrak{A}^+} / \mathfrak{N}_i)}{(\overline{\mathfrak{N}^+} / \mathfrak{N}_i)} .$$

Therefore we can identify $\mathscr{H}_{\text{phy}}$ with the closure of the set $\mathscr{H}_{\text{new}}/\mathfrak{N}$ in the metric $(\cdot, \cdot)$. Here

$$\mathfrak{N} = \{ f \in \mathscr{H}_{\text{new}} : (f, f) = 0 \}.$$

Since $\mathscr{T}$ is a non-negative self-adjoint operator, the square root $A = \mathscr{T}^{1/2}$ exists. The operator A is also nonnegative and self-adjoint on $\mathscr{H}_{\text{new}}$.

Furthermore, we have

$$(f, g) = (f, \mathscr{T} g)_{\text{new}} = (f, A^* A g)_{\text{new}} = (Af, Ag)_{\text{new}} .$$

Therefore A may be extended to a unitary mapping

$$A : \mathscr{H}_{\text{phy}} \to \mathscr{H}_{\text{new}} .$$

Lemma 8.2. *Under the mapping $A : \mathscr{H}_{\text{phy}} \to \mathscr{H}_{\text{new}}$ the operator $A \delta A^*$ corresponds to S.*

Proof. $(f, Sg) = (f, \mathscr{T} Sg)_{\text{new}} = (Af, (A \delta A) Ag)_{\text{new}}$. In other words, we have the following theorem.

Theorem 8.1. *The operator S^2 in $\mathscr{H}_{\text{phy}}$ is unitarily equivalent to the operator $A\delta\mathscr{F}\delta A^*$ in $\mathscr{H}_{\text{new}}$.*

We will study the operator S^2, since S is not positive definite.

9. Spectrum of S^2

By Theorem 8.1, we may forget S^2 and $\mathscr{H}_{\text{phy}}$ and instead study the operator $\mathscr{D} = A\delta\mathscr{F}\delta A^*$ in $\mathscr{H}_{\text{new}}$.

We say that the operator $\mathscr{D}_1$ is dual to $\mathscr{D}_2$ if there exists two bounded linear operators A and B such that $\mathscr{D}_1 = AB$ and $\mathscr{D}_2 = BA$. Our method of studying the spectrum and invariant subspaces will be based on the connection between the spectra of dual operators. This connection is described in the following lemma. We let $\sigma(A)$ denote the spectrum of A.

Lemma 9.1. *Suppose that A and B are bounded linear operators on the Hilbert space $\mathscr{H}$. Then*

$$\sigma(AB)\backslash\{0\} \equiv \sigma(BA)\backslash\{0\}. \tag{9.1}$$

If $\mathscr{H}'$ is a closed subspace invariant with respect to AB, the image $B\mathscr{H}'$ is invariant with respect to BA, and if the restriction of AB to $\mathscr{H}'$ has a (closed) spectrum that does not contain 0, the $B\mathscr{H}'$ is closed and

$$\alpha(AB|_{\mathscr{H}'}) = \sigma(BA|_{B\mathscr{H}'}). \tag{9.2}$$

Proof. Suppose that $\mu \in \sigma(AB)\backslash\{0\}$. Then there exists a sequence of vectors $\{f_n\}$ in $\mathscr{H}$ such that

$$\|f_n\| = 1, \qquad \|(AB-\mu)f_n\| \to 0.$$

Here it may be assumed that $\|Bf_n\| > c > 0$, since otherwise μ would be zero. Let us write

$$g_n = \frac{Bf_n}{\|Bf_n\|}\,.$$

Then $\|g_n\| = 1$ and

$$\|(BA-\mu)g_n\| = \frac{1}{\|Bf_n\|}\,\|B(AB-\mu)f_n\| \leqslant \frac{1}{c}\,\|B\|\,\|(AB-\mu)f_n\| \to 0.$$

This means that $\mu \in \sigma(BA)\backslash\{0\}$ or

$$\sigma(AB)\backslash\{0\} \subset \sigma(BA)\backslash\{0\}.$$

Because of symmetry, we obtain the first of these statements. Now suppose that $(AB)\mathscr{H}' \subset \mathscr{H}'$. Then

$$(BA)(B\mathscr{H}') = B((AB)\mathscr{H}') \subset B\mathscr{H}'.$$

Furthermore, note that if $\tilde{B}$ is the restriction of B to $\mathscr{H}'$, then $\tilde{B}^{-1}$ exists and is bounded in $B\mathscr{H}'$, since $0 \notin \sigma(B)$ (although if $0 \in \sigma(\tilde{B})$, we will have

$$0 \in \sigma(AB|_{\mathscr{H}'})).$$

Hence it follows that $B\mathscr{H}'$ is closed. Let us prove the latter assertion. Suppose that

$$0 \in \sigma(BA|_{B\mathscr{H}'}).$$

Then there exists a sequence $g_n \in B\mathscr{H}'$ such that

$$\|g_n\| = 1 \quad \text{and} \quad \|BAg_n\| \to 0.$$

Let us consider

$$\left\| AB \frac{B^{-1}g_n}{\|B^{-1}g_n\|} \right\| \leqslant \|B\| \, \|Ag_n\| \leqslant \|B\| \, \|\tilde{B}^{-1}\| \, \|BAg_n\| \to 0.$$

Hence $0 \in \sigma(AB|_{\mathscr{H}'})$. We have obtained a contradiction. The rest of the argument coincides with the proof of the first assertion. Lemma 9.1 is proved.

We now use the fact that the operators $\mathscr{D}_1 = \delta\mathscr{F}\delta\mathscr{F} = \delta\mathscr{F}\delta A^*A$ and $\mathscr{D}_2 = \mathscr{F}\delta\mathscr{F}\delta = A^*A\delta\mathscr{F}\delta$ are dual to $\mathscr{D}$ and to each other.

We first study the cluster properties of $\mathscr{D}_1$ and $\mathscr{D}_2$, i.e., we prove analogs of Lemmas 7.1 and 7.2.

Lemma 9.2. *In the basis* $\{f_I\}$, *the matrix elements of the operator* $\delta\mathscr{B}\delta\mathscr{B} = \mathscr{B}^2 = \mathscr{B}\delta\mathscr{B}\delta$ *satisfy the inequalities*

$$(c_1\lambda)^{2|I|} \leqslant (\mathscr{B}^2)_{II} \leqslant (c_2\lambda)^{2|I|}. \tag{9.3}$$

This lemma is an obvious corollary of Lemma 7.1.

Lemma 9.3. *Now suppose that* $a_{IJ}^{(i)}$, $i = 1, 2$, *are the matrix elements of* $\mathscr{D}_i - \mathscr{B}^2$. *Then the following cluster estimates hold*:

$$\sum_{I' : |I'| = m} |a_{II'}| \leqslant \begin{cases} (c\lambda)^{2m+1} & \text{if} \quad m = |I|; \\ (c\lambda)^{\max\{2m+1, |I|\}}, & m \neq |I|. \end{cases}$$

The proof follows from Lemma 7.2.

We now state the basic result on the spectrum of the transfer matrix.

Theorem 9.1. *For any integer* $N \geqslant 1$, *there exists a number* $\lambda_0 > 0$ *such that for all* $\lambda : 0 < \lambda < \lambda_0$, *the operator* S^2 *on* $\mathscr{H}_{\text{phy}}$ *has* $N + 2$ *mutually orthogonal invariant subspaces*:

$$\mathscr{H}_0 = \{\text{const}\}, \mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_N,$$

$$\overline{\mathscr{H}}_N = \mathscr{H}_{\text{phy}} \ominus \left(\bigoplus_{K=1}^{N} \mathscr{H}_K \right).$$

These subspaces are invariant under translation. The spectra $\sigma(S^2|_{\mathscr{H}_K})$ of the restrictions of S^2 to each of the subspaces satisfy the conditions

$$\begin{aligned} \sigma\left(S^2|_{\mathscr{H}_N}\right) &\subseteq \left[c_2\lambda^{2K}, c_1\lambda^{2K}\right], \qquad K=1,2,\ldots,N; \\ \sigma\left(S^2|_{\tilde{\mathscr{H}}_N}\right) &\subseteq \left[0, c_1\lambda^{2(N+1)}\right]. \end{aligned} \tag{9.5}$$

Proof. Using the cluster estimates from Lemmas 9.2 and 9.3, we may construct (for $i=1$ and 2) a set of subspaces

$$\mathscr{H}_0 = \{\text{const}\}, \mathscr{H}_1(i), \mathscr{H}_2(i), \ldots, \mathscr{H}_N(i), \tilde{\mathscr{H}}_N(i)$$

with the following properties:

1. mutual orthogonality;
2. $$\tilde{\mathscr{H}}_N(i) = \mathscr{H}_{\text{new}} \ominus \left(\bigoplus_{K=1}^{N} \mathscr{H}_K(i)\right);$$
3. $\mathscr{H}_K(i)$ are invariant with respect to the $\mathscr{D}_i$ and to translations in $\mathscr{H}_{\text{new}}$;
4. $$\begin{aligned} \sigma\left(\mathscr{D}_i|_{\mathscr{H}_K(i)}\right) &\subseteq \left[c_2\lambda^{2K}, c_1\lambda^{2K}\right], \qquad K=1,2,\ldots,N; \\ \sigma\left(\mathscr{D}_i|_{\tilde{\mathscr{H}}_N(i)}\right) &\subset \left[0, c_1\lambda^{2(N+1)}\right]. \end{aligned}$$

This construction is entirely analogous to the proof of the basic theorem in [9], and we will not discuss it here.

It now follows from the duality of $\mathscr{D}_1$ and $\mathscr{D}_2$ and from Lemma 9.1 that

$$\delta\mathscr{H}_K(2) \subset \mathscr{H}_K(1), \qquad \mathscr{F}\mathscr{H}_K(1) \subset \mathscr{H}_K(2), \qquad K=1,\ldots,N. \tag{9.6}$$

Hence,

$$\delta\mathscr{F}\mathscr{H}_K(1) \subset \delta\mathscr{H}_K(2) \subset \mathscr{H}_K(1).$$

But $\delta\mathscr{F}\mathscr{H}_K(1) = \mathscr{H}_K(1)$, because of the bounds for the spectrum. Therefore

$$\delta\mathscr{H}_K(2) = \mathscr{H}_K(1),$$

and similarly

$$\mathscr{F}\mathscr{H}_K(1) = \mathscr{H}_K(2).$$

To prove the theorem, we set

$$\mathscr{H}_K = A\mathscr{H}_K(1), \qquad K=1,2,\ldots,N.$$

We choose $\tilde{\mathscr{H}}_N$ as the orthogonal complement of the direct sum of $\mathscr{H}_0, \mathscr{H}_1, \ldots, \mathscr{H}_N$. By Lemma 9.1, we now have all the assertions of our theorem except perhaps the last assertion, i.e.,

$$\sigma(\mathscr{D}|_{\overline{\mathscr{H}}_N}) \subset \left[0, (c_1, \lambda)^{2(N+1)}\right]. \tag{9.7}$$

Suppose that there exists some invariant subspace $\mathscr{H}'$ of the operator $\mathscr{D}$ which is orthogonal to the subspace $\oplus_{K=0}^{N}\mathscr{H}_K$, and let

$$\sigma(\mathscr{D}|_{\mathscr{H}'}) \subset \left[(c_2\lambda)^{2N}, 1\right]. \tag{9.8}$$

Then again by Lemma 9.1 and the properties of the invariant subspaces of $\mathscr{D}_2$, we have

$$A^*\mathscr{H}' \subset \bigoplus_{K=0}^{N} \mathscr{H}_K(2) = \bigoplus_{K=0}^{N} \mathscr{F}\mathscr{H}_K(1).$$

Therefore $\mathscr{H}' \subset \oplus_{K=0}^{N}\mathscr{H}_K$, which follows from the explicit form of A^*.

10. Perturbation of the Dirac field

Here we construct a transformation by means of which it is possible to reduce a perturbation of an isotropic Dirac field to the nonisotropic case considered in Section 7.

For a Dirac field on a lattice, formal interaction may be taken in the form

$$U = \sum_t \psi^1(t)\psi^2(t) + \lambda \sum_{t-t'} \psi^1(t)\gamma(t-t')\psi^2(t'), \tag{10.1}$$

where

$$t, t' \in \tilde{Z}^4, \qquad \psi^1(t) = \left(\psi_1^1(t), \psi_2^1(t), \psi_3^1(t), \psi_4^1(t)\right)$$

is a row vector and $\psi^2(t)$ is the column vector formed by the generators of the Grassmann algebra

$$\gamma(t-t') = -\gamma(t'-t) = \begin{cases} \gamma_\mu^E & \text{if } \ t - t' = e_\mu, \quad \mu = 0,1,2,3; \\ 0 & \text{if } \ |t-t'| \neq 1. \end{cases}$$

Here the Euclidean Dirac matrices have the form

$$\gamma_0 = \gamma_0^E = \begin{vmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{vmatrix}, \qquad \gamma_1^E = \begin{vmatrix} 0 & & & i \\ & & i & \\ & -i & & \\ -i & & & 0 \end{vmatrix},$$

$$\gamma_2^E = \begin{vmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{vmatrix}, \qquad \gamma_3^E = \begin{vmatrix} 0 & & i & 0 \\ & & 0 & -i \\ -i & 0 & & \\ 0 & i & & 0 \end{vmatrix},$$

$$\left\{\gamma_\mu^E, \gamma_\nu^E\right\} = 2\delta_{\mu\nu}, \qquad \left(\gamma_\mu^E\right)^* = \gamma_\mu^E,$$

$$e_1 = (0,1,0,0), \qquad e_2 = (0,0,1,0), \qquad e_3 = (0,0,0,1).$$

The effect of (10.1) may be written concisely in the form

$$U = \psi^1(1 + \lambda_\gamma)\psi^2, \tag{10.2}$$

where $\gamma = \{\gamma(t - t')\}_{t,t'} \in \tilde{Z}^4$ is an infinite-dimensional matrix. We define $\gamma_{\text{space}} = \tilde{\gamma}_s$ as follows:

$$\tilde{\gamma}_s(t - t') = \begin{cases} \gamma(t - t') & \text{if } t^0 = t^{0\prime}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{\gamma}_0 = \gamma - \tilde{\gamma}_s$; then

$$\tilde{\gamma}_0\tilde{\gamma}_s = -\tilde{\gamma}_s\tilde{\gamma}_0, \tag{10.3}$$

as may be verified by simple computation. We introduce new Grassmann generators

$$\varphi^1 = \psi^1(1 + \lambda\tilde{\gamma}_s/2), \qquad \varphi^2 = (1 + \lambda\tilde{\gamma}_s/2)\psi^2.$$

Then

$$\psi^1 = \varphi^1(1 + \lambda\tilde{\gamma}_s/2)^{-1}, \qquad \psi^2 = (1 + \lambda\tilde{\gamma}_s/2)^{-1}\varphi^2. \tag{10.4}$$

The inverse of the matrix $(1 + \lambda\tilde{\gamma}_s/2)$ evidently exists for the small λ (for example, it may be an operator in $\mathfrak{A}$). Substituting (10.4) into (10.2) yields

$$U = \varphi^1(1 + \lambda\tilde{\gamma}_s/2)^{-1}(1 + \lambda\tilde{\gamma}_s + \lambda\tilde{\gamma}_0)(1 + \lambda\tilde{\gamma}_s/2)^{-1}\varphi^2.$$

Furthermore, we readily observe that

$$(1 + \lambda\tilde{\gamma}_s/2)^{-1}(1 + \lambda\tilde{\gamma}_s)(1 + \lambda\tilde{\gamma}_s/2)^{-1} = 1 + \tilde{\tilde{\gamma}},$$

where

$$\|\tilde{\tilde{\gamma}}(t - t')\| \leqslant \min\left\{(c\lambda)^2, (c\lambda)^{|t - t'|}\right\}$$

(norm in $\mathfrak{A}$). Furthermore,

$$\begin{aligned}
&(1 + \lambda\tilde{\gamma}_s/2)^{-1}\lambda\tilde{\gamma}_0(1 + \lambda\tilde{\gamma}_s/2)^{-1} \\
&= \sum_{K=0}^{\infty}(-\tfrac{1}{2}\lambda\tilde{\gamma}_s)^K\lambda\tilde{\gamma}_0\sum_{K=0}^{\infty}(-\tfrac{1}{2}\lambda\tilde{\gamma}_s)^K \\
&= \sum_{K=0}^{\infty}\sum_{l=0}^{\infty}\lambda(-\tfrac{1}{2}\lambda)^{K+l}\tilde{\gamma}_s^K\tilde{\gamma}_0\tilde{\gamma}_s^l \\
&= \lambda\tilde{\gamma}_0 + \lambda\sum_{K=0}^{\infty}\sum_{l=0}^{\infty}(-\tfrac{1}{2}\lambda)^{K+l+1}\tilde{\gamma}_s^K\left[\tilde{\gamma}_s\tilde{\gamma}_0 + \tilde{\gamma}_0\tilde{\gamma}_s\right]\tilde{\gamma}_s^l \\
&\quad -\lambda\sum_{K=1}^{\infty}\sum_{l=1}^{\infty}(-\tfrac{1}{2}\lambda)^{K+l}\tilde{\gamma}_s^K\tilde{\gamma}_0\tilde{\gamma}_s^l \\
&= \lambda\tilde{\gamma}_0 + 0 + \tilde{\tilde{\tilde{\gamma}}},
\end{aligned}$$

since the expression in brackets vanishes by (10.3).

Thus (10.2) may be rewritten in the form

$$U = \varphi^1(1 + \lambda\tilde{\gamma}_0 + \tilde{\tilde{\gamma}} + \tilde{\tilde{\tilde{\gamma}}})\varphi^2 + \sum_t \varphi^1(t)\varphi^2(t)$$

$$+ \lambda \sum_t \left\{ \varphi^1(t)\gamma_0\varphi^2(t + e_0) - \varphi^1(t + e_0)\gamma_0\varphi^2(t) \right\}$$

$$+ \sum_{t,t'} \varphi^1(t)(\tilde{\tilde{\gamma}}(t - t') + \tilde{\tilde{\tilde{\gamma}}}(t - t'))\varphi^2(t').$$

After the substitutions

$$\varphi'_\alpha(t) \mapsto i\varphi^2_\alpha(t), \qquad \varphi^2_\alpha(t) \mapsto i\varphi^1_\alpha(t); \qquad \alpha = 3, 4,$$

we obtain

$$U = \sum_t \sum_\alpha \varphi^1_\alpha(t)\varphi^2_\alpha(t) + \lambda \sum_t \sum_\alpha \sum_\varepsilon \varphi^\varepsilon_\alpha(t)\varphi^{3-\varepsilon}_\alpha(t + e_0)$$

$$+ \sum_A c_A \psi_A ,$$

where

$$|c_A| < \min\left\{ (c\lambda)^2, (c\lambda)^{|t - t'|} \right\},$$

i.e., c_A satisfies the bounds postulated in Section 7.

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