MULTIPLICATIVE AND ADDITIVE CLUSTER EXPANSIONS FOR THE EVOLUTION OF QUANTUM SPIN SYSTEMS IN THE GROUND STATE

V.A. MALYSHEV and R.A. MINLOS
Moscow State University, Moscow, USSR

Received 15 July 1981

It is shown that for the ν-dimensional quantum Ising model in the high temperature region e^{-tH} in the GNS representation admits a "multiplicative" N-particle cluster expansion and H admits an "additive" N-particle cluster expansion.

Let us consider the $(\nu+1)$-dimensional Ising model with continuous time. It is a random field on $Z^\nu \times R = \{(x,t): x \in Z^\nu, t \in R\}$ with values ± 1 in each point. To define it we consider first for any $x \in Z^\nu$ the stationary Markov process $\xi_x(t)$ with two values ± 1 defined by the stochastic semigroup $\exp(-tH_{0x})$, $H_{0x} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$. For different points x the processes $\xi_x(t)$ are mutually independent. We denote by (Ω, Σ, μ_0) the probability measure space where all $\xi_x(t)$ are defined. Ω can be identified with the set of all functions $f(x,t)$ with values ± 1 which are stepwise constant for any fixed x.

Let us consider a new probability measure $\mu_{\Lambda,T}$ on (Ω, Σ) with density

$$\frac{d\mu_{\Lambda,T}}{d\mu_0} = Z^{-1}_{\Lambda,T} \exp\left(\beta \sum_{|x-x'|=1} \int_{-T}^{T} \xi_x(t)\, \xi_{x'}(t)\, dt\right)$$

where Λ is the cube in Z^ν, $x, x' \in \Lambda$, $T > 0$.

We shall consider only the case when β is sufficiently small, $|\beta| \leqslant \beta_0(\lambda)$. It is a standard result then that there exists a weak limit for $\Lambda \uparrow Z^\nu$, $T \to \infty$: $\mu = \lim \mu_{\Lambda,T}$.

Let for any $A \subset Z^\nu \times R$ Σ_A be the minimal σ algebra with regard to which all $\xi_x(t)$ are measurable for $(x,t) \in A$. The physical Hilbert space is defined as

$$\mathcal{H} = L_2(\Omega, \Sigma_0, \mu), \quad \Sigma_0 = \Sigma_{Z^\nu \times \{0\}} .$$

The stochastic semigroup $\mathcal{T}_t: \mathcal{H} \to \mathcal{H}$ (transfer matrix) is defined by its matrix elements

$$(\xi_1, \mathcal{T}_t \xi_2) = \langle \xi_1 (U_t \xi_2) \rangle ,$$

where $\langle\ \rangle = \langle\ \rangle_\mu$, $\xi_1, \xi_2 \in \mathcal{H}$, U_t is the translation from the slice $Z^\nu \times \{0\}$ onto the slice $Z^\nu \times \{t\}$.

One can show that $\mathcal{T}_t$ is the strongly continuous semigroup of positive self-adjoint operators and $\|\mathcal{T}_t\| = 1$. Then $\mathcal{T}_t = e^{-tH}$ where H is positive self-adjoint.

The unitary group e^{-itH} can then be identified [1] with the evolution of a ν-dimensional quantum spin system with the formal hamiltonian

$$H_{\text{formal}} = 2\lambda \sum_x \sigma_x^{(1)} + \beta \sum_{|x-x'|=1} \sigma_x^{(3)} \sigma_{x'}^{(3)} .$$

Then $\mathcal{H}$ is the space of the GNS representation of the quasilocal C^* algebra in the ground state.

We shall obtain a cluster expansion for e^{-tH} and H on the special basis (first appeared in ref. [2]) which we shall now define.

Let T_x, $x \in Z^\nu$, be the set of all $y \in Z^\nu$ such that $y < x$ in lexicographic order. P_x is the orthogonal projection in $L_2(\Omega, \Sigma_0, \mu)$ onto $L_2(\Omega, \Sigma_{T_x}, \mu)$. Let us put

$$\hat{\xi}_x = \hat{\xi}_x(0) = \xi_x(0) - P_x \xi_x(0), \quad f_x = \hat{\xi}_x (P_x \hat{\xi}_x^2)^{-1/2},$$

$$f_I = \prod_{x \in I} f_x, \quad I \subset Z^\nu, \quad |I| < \infty, \quad f_\emptyset = 1,$$

$$f_x(t) = U_t f_x, \quad f_I(t) = \prod_{x \in I} f_x(t).$$

Theorem 1. $\{f_I\}$ is a complete orthonormal basis in $\mathcal{H}$. Moreover

0 031-9163/81/0000–0000/$ 02.75

$$\langle f_I \rangle = 0, \quad I \neq \emptyset, \quad M(f_x/\Sigma_{T_x}) = 0, \quad M(f_x^2/\Sigma_{T_x}) = 1. \tag{2}$$

Theorem 2. The semi-invariants

$$\omega(I, I'; t) = \langle f_{x_1}(0), ..., f_{x_n}(0), f_{x'_1}(t), ..., f_{x'_m}(t) \rangle,$$

$$I = \{x_1, ..., x_m\}, \quad I' = \{x'_1, ..., x'_m\},$$

are C^∞ in t, $t \geq 0$, and for $I, I' \neq \emptyset$

$$\left| \frac{d^k \omega(I, I'; t)}{dt^k} \right| \leq (C\beta)^{d^1} (Ce^{-\lambda})^{d^2}, \quad k = 0, 1, 2, \ldots . \tag{3}$$

$\omega(I, I'; t) = 0$ if I or I' is empty.

Here $I'(t)$ is the translation of I' to the slice $Z^\nu \times \{t\}$, $d^{1(2)}$ is the minimal sum of the lengths of the bonds of any connected tree with vertices in $I \cup I'(t)$ in the metrics $|y^1| + |y^2| + \cdots |y^{\nu+1}|$ in $Z^\nu \times R$, the Z^ν direction (R direction) C depends only upon ν and k.

The proof of this theorem is similar to the proof of the same theorem for discrete time models in ref. [3] and will appear in ref. [4].

Multiplicative cluster expansion for e^{-tH}.

Theorem 3. The matrix elements of e^{-tH} admit the following representation for $I, I' \neq \emptyset$:

$$(f_I, e^{-tH} f_{I'}) = \sum \omega(I_1, I'_1; t) \cdots \omega(I_n, I'_n; t) \tag{4}$$

and are equal to 0 if $I = \emptyset$ or $I' = \emptyset$. The sum is over all partitions

$$I_1 \cup ... \cup I_n = I, \quad I'_1 \cup ... \cup I'_n = I' .$$

Proof. This follows from (1) and the formula which expresses moments through the semi-invariants.

Additive cluster expansion for H.

Theorem 4. Matrix element of H are equal to

$$(f_I, Hf_{I'}) = -\sum \omega'(I_1, I'_1; 0), \tag{5}$$

where the sum is over all nonempty $I_1 \subset I, I'_1 \subset I'$ such that $I - I_1 = I' - I'_1$.

Moreover $\omega'(I, I'; 0) \equiv d\omega(I, I'; 0)/dt$ satisfies the estimates (3) with $d^2 = 0$.

Proof. Let us calculate the derivative for $t = 0$ on both sides of (4). Let us note then that

$$\omega(I, I'; 0) = 1, \quad \text{if } I = I' = \{x\}, \qquad = 0, \quad \text{in other cases.} \tag{6}$$

In fact we have

$$\langle f_I f_{I'} \rangle = 0, \quad I \neq I', \tag{7}$$

if we use (2) and $M\eta = M(M(\eta/\Sigma_{T_{0,x}}))$. Also we have

$$\langle f_I^2 \rangle = 1. \tag{8}$$

Using the Möbius inversion formula we get from (7) that $\omega(I, I'; 0) = 0$ if $I \neq I'$. The first part of (6) follows from (2). Finally

$$\omega(I, I; 0) = \sum_k \sum (-1)^{k-1}(k-1)! = 0,$$

where the sum is over all partitions of $\{1, ..., n\}$, $n = |I| \geq 2$.

This gives the complete N-particle cluster expansions for all N and $|\beta| \leq \beta_0(\lambda)$.

The present result can be extended to other lattice models in high- and low-temperature regions (see e.g. ref. [3]). This and applications to the study of the spectral properties of H will appear in forthcoming articles.

We remark that all ω can be represented as an explicit series with exponential convergence.

References

[1] V.A. Malyshev, Funktionalnyi Analyz i Jogo Pril. 13 (1979) 31.
[2] R.A. Minlos and Ja.G. Sinai, Teor. Mat. Phys. 2 (1970) 230.
[3] V.A. Malyshev, Usp. Mat. Nauk. 35 (1980) 3.
[4] V.A. Malyshev and R.A. Minlos, Trudy Petrovskii Seminar (Moscow), to be published.