When bounded chaos becomes unbounded

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Uniformly Bounded Initial Chaos in Large System Often Intensifies Infinitely

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Abstract

We consider infinite harmonic chain with completely deterministic dynamics. Initial data are assumed absolutely bounded. Nevertheless maximum of the variables can grow infinitely in time. We give conditions for this phenomenon. It coincides with intuitive guess that the main condition for this growth is sufficient chaos in the initial conditions.

1 Introduction

Our goal is to study models of various qualitative phenomena in non equilibrium infinite particle systems. Normally such models use stochastic dynamics. The goal of our project is to show that completely different approach could be chosen – with minimum possible probability. Here we consider the simplest deterministic example of such models. We assume that initially the deviations from equilibrium are uniformly bounded. Could they grow in time and how? The same problem of course exists for finite but large number of particles, but it demands some scaling for time, number of particles etc. In recent papers [2, 3, 4] we considered problems related to convergence to equilibrium for finite number of particles.

We consider trajectories $x_k(t), k \in Z$, for standard countable linear chain of particles defined by the formal interaction energy

$$U = \frac{\omega_0^2}{2}\sum_k (x_k - a_k)^2 + \frac{\omega_1^2}{2}\sum_k (x_{k+1} - x_k - (a_{k+1} - a_k))^2$$

where $\omega_1 > 0$, and

$$\ldots < a_k < a_{k+1} < \ldots$$

where $a_k \to \pm\infty$ if $k \to \pm\infty$. Normally only the case when $a_k = ka$ for some a, is considered but we will see that it is almost the same. What is more important, we consider here only the case when $\omega_0 = 0$, seemingly in case $\omega_0 > 0$ less chaos is expected as any particle is tightly bounded to a fixed place. This case will be considered elsewhere.

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If we introduce deviations $q_k = x_k - a_k$, then the energy can be written as

$$U = \frac{\omega_1^2}{2} \sum_k (q_{k+1} - q_k)^2.$$

The equilibrium corresponds to $q_k \equiv 0$, and we consider the equations

$$\frac{d^2 q_k}{dt^2} = \omega_1^2 (q_{k+1} - 2q_k + q_{k-1}) = \omega_1^2 (\Delta q)_k \tag{1}$$

with some initial conditions $q_k(0), \dot{q}_k(0)$.

2   Results

Remind the following two spaces of functions on $\mathbb{Z}$:

$$l_\infty = l_\infty(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{R} : \sup_{k \in \mathbb{Z}} |f(k)| < \infty\}, \quad |f|_\infty = \sup_{k \in \mathbb{Z}} |f(k)|,$$

$$l_2 = l_2(\mathbb{Z}) = \left\{f : \mathbb{Z} \to \mathbb{R} : \sum_{k \in \mathbb{Z}} |f(k)|^2 < \infty\right\}, \quad |f|_2 = \sqrt{\sum_{k \in Z} |f(k)|^2}.$$

Put $p(t) = \dot{q}(t)$.

Proposition 2.1. *Assume $q(0) \in l_2(\mathbb{Z})$, $p(0) = 0$, then*

$$|q(t)|_\infty \leqslant |q(0)|_2.$$

Thus here the solution is uniformly bounded (in particle's index k and time t). The situation drastically changes if the initial conditions belong to l_∞. The next statements show that growth cannot exceed the order $\sqrt{t}$ and that there exist initial conditions with this order of growth.

Theorem 2.1. *The following statements hold:*

1. *Let $q(0) \in l_\infty(\mathbb{Z})$, $p(0) = 0$, then for any $t \geqslant 0$ the following inequality holds:*

$$|q(t)|_\infty \leqslant \left(\sqrt{2\gamma\omega_1 t} + 2\right)|q(0)|_\infty,$$

where $\gamma > 0$ is the unique positive root of the equation:

$$\frac{1}{\gamma}e^{1/\gamma} = \frac{1}{e}.$$

2. *For any $n \in \mathbb{Z}$ there exist: a constant $c > 0$, initial conditions $q(0) \in l_\infty(\mathbb{Z})$, $p(0) = 0$ and an increasing sequence of time moments $t_1 < t_2 < \ldots$, $t_k \to \infty$ as $k \to \infty$, such that*

$$q_n(t_{2k}) \geqslant c\sqrt{t_{2k}}, \quad q_n(t_{2k+1}) \leqslant -c\sqrt{t_{2k+1}}$$

for any $k = 1, 2, \ldots$

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Corollary 2.1. *For any $n \in \mathbb{Z}$ there exist initial conditions $q(0) \in l_\infty(\mathbb{Z})$, $p(0) = 0$ such that*

$$\limsup_{t \to \infty} \frac{q_n(t)}{\sqrt{t}} = c_1 > 0, \quad \liminf_{t \to \infty} \frac{q_n(t)}{\sqrt{t}} = c_2 < 0$$

for some constants c_1, c_2.

Next theorem claims that if the initial conditions are sufficiently random, then any $q_n(t)$ can be arbitrary big with t.

Theorem 2.2. *Suppose that $q_k(0)$, $k \in \mathbb{Z}$, is a sequence of i.i.d. random variables such that $Eq_k(0) = 0, Eq_k^2(0) > 0$ and $E|q_0(0)|^3 < \infty$. Then for all $n \in \mathbb{Z}$ the following formula holds*

$$P\{\sup_{t \geqslant 0} q_n(t) = +\infty\} = P\{\inf_{t \geqslant 0} q_n(t) = -\infty\} = 1, \tag{2}$$

where by $P(\cdot)$ we denote the probability of the corresponding event.

Next theorem explains in some way the formal nature of the previous results.
The following operator on l_∞

$$(Vq)_k = -\omega_1^2(\Delta q)_k = -\omega_1^2(q_{k+1} - 2q_k + q_{k-1})$$

is bounded, namely $|V|_\infty \leqslant 4\omega_1^2$. Then the following operator is also bounded on l_∞:

$$C(t) = \cos(t\sqrt{V}) = \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}V^k}{(2k)!}. \tag{3}$$

From (the proofs of) theorems 2.1 and 3.1 the following statement follows.

Theorem 2.3. *There exist constants $a, b > 0$ such that for all $t \geqslant 0$ the following inequalities hold:*

$$a\sqrt{t} + 1 \leqslant \left|\cos(t\sqrt{V})\right|_\infty \leqslant b\sqrt{t} + 1.$$

3 Proofs

3.1 Various expressions for the dynamics

Lemma 3.1. *The solution of the main system (1) can be written as*

$$q_n(t) = \sum_{k \in \mathbb{Z}} a_k(t)q_{n-k}(0) + \sum_{k \in \mathbb{Z}} b_k(t)p_{n-k}(0), \tag{4}$$

where

$$a_k(t) = \frac{1}{2\pi}\int_0^{2\pi}\cos\left(2\omega_1 t \sin\frac{\lambda}{2}\right)e^{ik\lambda} \, d\lambda,$$

$$b_k(t) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin\left(2\omega_1 t \sin(\lambda/2)\right)}{2\omega_1 \sin(\lambda/2)}e^{ik\lambda} \, d\lambda.$$

Proof. Let us prove that infinite vectors $a = \{a_k\}$ and b satisfy the system (1). We have

$$\ddot{a}_k(t) = -\frac{\omega_1^2}{2\pi} \int_0^{2\pi} \left(2\sin\frac{\lambda}{2}\right)^2 \cos\left(2\omega_1 t \sin\frac{\lambda}{2}\right) e^{ik\lambda} \, d\lambda.$$

From

$$2\sin^2\frac{\lambda}{2} = 1 - \cos\lambda = 1 - \frac{e^{i\lambda} + e^{-i\lambda}}{2}$$

it follows

$$\ddot{a}_k(t) = -\omega_1^2(2a_k(t) - a_{k+1}(t) - a_{k-1}(t)) = \omega_1^2(\Delta a)_k.$$

Similarly for b. Moreover,

$$a_k(0) = \dot{b}_k(0) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

It follows that the series in (4) formally satisfies the system (1) with corresponding initial conditions. It remains to prove that it is absolutely convergent and defines the solution from l_∞. Integrating by parts we get:

$$2\pi a_k(t) = \frac{1}{ik} \cos\left(2\omega_1 t \sin\frac{\lambda}{2}\right) e^{ik\lambda}\Big|_0^{2\pi} + \frac{1}{ik} \int_0^{2\pi} \omega_1 t \cos\frac{\lambda}{2} \sin\left(2\omega_1 t \sin\frac{\lambda}{2}\right) e^{ik\lambda} \, d\lambda$$

$$= \frac{\omega_1 t}{ik} \int_0^{2\pi} \cos\frac{\lambda}{2} \sin\left(2\omega_1 t \sin\frac{\lambda}{2}\right) e^{ik\lambda} \, d\lambda$$

$$= \frac{\omega_1 t}{k^2} \int_0^{2\pi} \frac{d}{d\lambda}\left(\cos\frac{\lambda}{2} \sin\left(2\omega_1 t \sin\frac{\lambda}{2}\right)\right) e^{ik\lambda} \, d\lambda.$$

It follows that $|a_k(t)| \leqslant c/k^2$ for some constant c not depending on k. Similar bound exists for b_k. Thus the series in (4) are uniformly bounded by $c(t)|q(0)|_\infty$ for some function $c(t)$. The lemma is thus proved. $\qquad\square$

We will need the following integral presentation for Bessel functions of integer index:

$$J_n(t) = \frac{1}{\pi} \int_0^{\pi} \cos(t \sin\phi - n\phi) \, d\phi.$$

Lemma 3.2. *Let $p(0) = 0$, then we have:*

$$q_n(t) = \sum_k^{\infty} J_{2k}(2\omega_1 t) q_{n-k}(0). \tag{5}$$

Proof. We use the equality

$$a_k(t) + a_{-k}(t) = \frac{1}{\pi} \int_0^{2\pi} \cos\left(2\omega_1 t \sin\frac{\lambda}{2}\right) \cos(k\lambda) \, d\lambda$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos\left(2\omega_1 t \sin\phi\right) \cos(2k\phi) \, d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\cos\left(2\omega_1 t \sin\phi + 2k\phi\right) + \cos\left(2\omega_1 t \sin\phi - 2k\phi\right)\right) d\phi$$

$$= J_{2k}(2\omega_1 t) + J_{-2k}(2\omega_1 t) = 2J_{2k}(2\omega_1 t).$$

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In the last equality we exploited the relation $J_{2k}(t) = J_{-2k}(t)$. Using formula (4) and the fact that $a_k(t) - a_{-k}(t)$ is either zero or pure imaginary, we can get

$$q_n(t) = \sum_k \left(\frac{1}{2}(a_k(t) + a_{-k}(t)) + \frac{1}{2}(a_k(t) - a_{-k}(t)) \right) q_{n-k}(0)$$

$$= \sum_k \frac{1}{2}(a_k(t) + a_{-k}(t)) q_{n-k}(0) = \sum_k^{\infty} J_{2k}(2\omega_1 t) q_{n-k}(0)$$

Lemma 3.2 is proved. $\qquad\square$

Note that (5) is the Neumann series (see [7]) for the solution $q_n(t)$ with coefficients determined by initial conditions.

Lemma 3.3. *The following formula holds:*

$$q(t) = C(t)q(0) + S(t)p(0),$$

where $C(t)$ is defined in (3), and for any t

$$S(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} V^k$$

is the operator in l_∞, continuous in t.

Proof. It is clear that the power series in t for $C(t)$ and $S(t)$ can be differentiated term by term, and we have the equalities:

$$\ddot{C}(t) = \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k-2} V^k}{(2k-2)!} = -VC(t), \quad C(0) = E,$$

$$\ddot{S}(t) = \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k-1}}{(2k-1)!} V^k = -VS(t), \quad S(0) = 0,$$

where E is the unit operator. $\qquad\square$

Lemma 3.4. *Let $q(0) \in l_\infty$, $p(0) = 0$, and moreover $q_n(0) = 0$ for all $|n| < M$ for some M. Then for any $t \geqslant 0$ the following inequality holds:*

$$|q_0(t)| \leqslant \left(e^{\alpha+1} \alpha \right)^{2M} |q(0)|_\infty, \quad \alpha = \omega_1 \frac{t}{M}.$$

Proof. This statement easily follows from formula (5) and classical bounds for the Bessel functions (for example, from Poisson's integral for the Bessel function, [7]). But we will give a proof without using a presentation of solution via the Neumann series. So our proof can be easily generalized to more common quadratic interaction potential.

We estimate $q(t)$ using lemma 3.3. We have:

$$q(t) = q(0) + \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} V^k q(0).$$

For any $k \geqslant 1$ the following holds:

$$(V^k q(0))_0 = \sum_{i_1, i_2, \ldots, i_k \in \mathbb{Z}} V_{i_0, i_1} V_{i_1, i_2} \ldots V_{i_{k-1}, i_k} q_{i_k}(0), \quad i_0 = 0.$$

If $(V^k q(0))_0 \neq 0$ for some k, then one can find a sequence $i_1, \ldots, i_k$ of indices such that $|i_r - i_{r-1}| \leqslant 1$ for all $r = 1, \ldots, k$ and $|i_k| \geqslant M$. Thus

$$M \leqslant |i_k| = \left| \sum_{r=1}^{k} i_r - i_{r-1} \right| \leqslant \sum_{r=1}^{k} |i_r - i_{r-1}| \leqslant k.$$

It follows that

$$q_0(t) = q_0(0) + \sum_{k=M}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} (V^k q(0))_0.$$

Then for the operator norm we have

$$|V^k q(0)|_\infty \leqslant |V|_\infty^k |q(0)|_\infty \leqslant (4\omega_1^2)^k |q(0)|_\infty.$$

The following inequalities hold:

$$|q_0(t)| \leqslant \sum_{k=M}^{\infty} \frac{(2\omega_1 t)^{2k}}{(2k)!} |q(0)|_\infty \leqslant e^{2\omega_1 t} \frac{(2\omega_1 t)^{2M}}{(2M)!} |q(0)|_\infty.$$

Since for any integer n

$$n! \geqslant \left(\frac{n}{e} \right)^n,$$

we have

$$|q_0(t)| \leqslant \left(\frac{\exp\{\omega_1 t/M\} \omega_1 t e}{M} \right)^{2M} |q(0)|_\infty.$$

The lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Proposition 2.1 As the function $\cos(2\omega_1 t \sin(\lambda/2))$ is infinitely smooth in λ and periodic with period 2π, we have

$$\cos\left(2\omega_1 t \sin\frac{\lambda}{2} \right) = \sum_{k \in \mathbb{Z}} a_k(t) e^{-i\lambda k}$$

for all $\lambda \in [0, 2\pi]$ and all $t \geqslant 0$. By formula (4) and Parseval's equality

$$|q_n(t)| \leqslant \sqrt{\sum_k |a_k(t)|^2} \sqrt{\sum_k q_k^2(0)}$$

$$= |q(0)|_2 \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \cos^2\left(2\omega_1 t \sin\frac{\lambda}{2} \right) d\lambda} \leqslant |q(0)|_2.$$

3.2 Proof of Theorem 2.1

3.2.1 Upper bound

Equivalent statement is that for all $t \geqslant 0$ and all $n \in \mathbb{Z}$ the inequality

$$|q_n(t)| \leqslant \left(\sqrt{2\gamma\omega_1 t} + 2 \right) |q(0)|_\infty$$

holds. Without loss of generality we can put $n = 0$. The idea of the proof is the following. Write the initial condition as the sum of two terms. The first one is the restriction of $q(0)$ on finite interval of $\mathbb{Z}$ containing 0 and having length of order t. The second one is the restriction of $q(0)$ on the remaining part of $\mathbb{Z}$. By Proposition 2.1 the solution corresponding to the first term is of the order $\sqrt{t}$. It remains to show that the influence of the second term for time of the order t is small. Now the formal proof.

Fix some time moment $T > 0$ and a constant $\mu > 0$. Write the initial vector as:

$$q(0) = q^2(0) + q^\infty(0),$$

where

$$q_n^2(0) = \begin{cases} q_n(0), & |n| \leqslant \mu T, \\ 0, & |n| > \mu T, \end{cases}$$

$$q_n^\infty(0) = \begin{cases} 0, & |n| \leqslant \mu T, \\ q_n(0), & |n| > \mu T. \end{cases}$$

Let $q^2(t), q^\infty(t)$ be the solutions of the system (1) corresponding to the initial conditions $q^2(0), p^2(0) = 0$ and $q^\infty(0), p^\infty(0) = 0$. For all $t \geqslant 0$:

$$q(t) = q^2(t) + q^\infty(t).$$

Since $q^2(0) \in l_2$, by Proposition 2.1 the following inequality holds for all $t \geqslant 0$:

$$|q^2(t)|_\infty \leqslant |q^2(0)|_2 \leqslant |q(0)|_\infty \sqrt{2\mu T + 1}.$$

By Lemma 3.4 we have also the inequality:

$$|q_0^\infty(t)| \leqslant \left(e^{\alpha(t)+1} \alpha(t) \right)^{2m(T)} |q(0)|_\infty,$$
$$\alpha(t) = \omega_1 \frac{t}{m(T)}$$

where $m(T) = [\mu T] + 1$, and $[x]$ is the integer part of x. Since $\mu T/m(T) \leqslant 1$ for all $T \geqslant 0$, we have

$$\alpha(T) \leqslant \frac{\omega_1}{\mu}.$$

It follows

$$e^{\alpha(T)+1} \alpha(T) \leqslant e^{\omega_1/\mu} \frac{\omega_1}{\mu} e.$$

Put $\mu = \gamma\omega_1$, where γ satisfies the equation

$$e^{1/\gamma}\frac{1}{\gamma}e = 1$$

(and thus it is uniquely defined). For given μ the following inequality holds:

$$|q_0^\infty(T)| \leqslant |q(0)|_\infty.$$

Thus, we have proved that for any $T \geqslant 0$ and $\mu = \gamma\omega_1$, the following inequality holds:

$$|q(T)|_\infty \leqslant (\sqrt{2\mu T + 1} + 1)|q(0)|_\infty \leqslant (\sqrt{2\gamma\omega_1 T} + 2)|q(0)|_\infty.$$

The proof is finished.

3.2.2   Lower bound

Like in the proof of item 1, without loss of generality we take $n = 0$. If $q \in l_\infty$, then define the support of q as follows:

$$\text{supp}(q) = \{n \in \mathbb{Z}: \ q_n \neq 0\}.$$

The idea of the proof is also simple. Firstly we prove that for any T there exists initial vector having support of the order T, and moreover such that maximum of the corresponding solution is bounded from below by $c\sqrt{T}$ for some c, not depending on T. Then, for some increasing sequence of T_k we sum up these initial conditions so that their supports do not intersect. Now the formal proof.

Theorem 3.1. *There is T_0 such that for any $T > T_0$ there are initial conditions $q(0) \in l_\infty$, $|q(0)|_\infty \leqslant 1, p(0) = 0$ with the following properties:*

1. $q_0(T) \geqslant c\sqrt{T}$ for some constant $c > 0$ not depending on T;

2. $\text{supp}(q(0)) \subset [aT, bT]$ for some positive constants $a, b > 0$ not depending on T.

Proof. For this proof we need some lemmas. For the number $\mu \in \mathbb{R}$ define the functions:

$$J(t, \mu) = \frac{1}{\pi} \int_0^\pi \cos(t(\sin\phi - \mu\phi)) \, d\phi.$$

Lemma 3.5. *For all $|\mu| < 1$ the following asymptotic formula holds as $t \to \infty$:*

$$J(t, \mu) = \sqrt{\frac{2\pi}{t\sqrt{1 - \mu^2}}} \cos\left(tg(\mu) - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{t}\right)\right)$$

where $O()$ is uniform in μ on any segment $[a, b] \subset (-1, 1)$ and

$$g(\mu) = \sqrt{1 - \mu^2} - \mu\arccos(\mu).$$

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Proof. This assertion is a well-known result about the Bessel functions (see [1, 7]). Nevertheless we will write the proof for completeness. We use stationary phase method for the function $J(t, \mu)$ in case when the phase function contains additional parameter ([1], p. 107, theorem 1.6). In our case the phase function is:

$$S(\phi, \mu) = \sin \phi - \mu \phi.$$

We have the following equalities for the derivatives:

$$S_\phi(\phi, \mu) = \cos \phi - \mu, \quad S_{\phi\phi}(\phi, \mu) = -\sin \phi.$$

It follows that S has the unique critical point on $[0, \pi]$, which for $|\mu| < 1$ is equal to

$$\phi_0 = \phi_0(\mu) = \arccos(\mu).$$

Then

$$S_{\phi\phi}(\phi_0(\mu), \mu) = -\sqrt{1 - \mu^2}.$$

It follows that $\phi_0(\mu)$ is a non-degenerate stationary point for $\mu \in [-1 + \delta, 1 - \delta]$ for any $0 < \delta < 1$. Thus all conditions of the mentioned theorem hold together with the following formula:

$$J(t, \mu) = \frac{1}{\pi} \sqrt{\frac{2\pi}{t\sqrt{1 - \mu^2}}} \cos\left(tS(\phi_0(\mu), \mu) - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{t}\right)\right)$$

as $t \to \infty$, where $O()$ is uniform in $\mu \in [a, b]$ for any segment $[a, b] \subset (-1, 1)$. We have the equality:

$$S(\phi_0(\mu), \mu) = \sqrt{1 - \mu^2} - \mu \arccos(\mu).$$

So the lemma is proved. $\qquad\square$

From the proven lemma it follows that, if $\mu t = k \in \mathbb{Z} > 0$ for some μ such that $\mu < 1/2$, then:

$$J_{2k}(t) = J(t, 2\mu) = \sqrt{\frac{2}{\pi t \sqrt{1 - 4\mu^2}}} \cos\left(tg(2\mu) - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{t}\right)\right), \quad \mu = \frac{k}{t}$$

as $t \to \infty$, where $O()$ is uniform in all k belonging to the segment $[at, bt]$ for all

$$0 \leqslant a < b < \frac{1}{2}.$$

Denote the main term in this asymptotic formula for $J_{2k}(t)$ by $f_k(t)$:

$$f_k(t) = \sqrt{\frac{2}{\pi t \sqrt{1 - 4\mu^2}}} \cos\left(tg(2\mu) - \frac{\pi}{4}\right), \quad \mu = \frac{k}{t}.$$

Then

$$J_{2k}(t) = f_k(t)\left(1 + O\left(\frac{1}{t}\right)\right), \tag{6}$$

where $O()$ has the properties as above.

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Lemma 3.6. *There exist numbers $0 < a < b < 1/2$ and $t_0 > 0$ such that for any $t \geqslant t_0$ there is a subset $I \subset ([at, bt] \cap \mathbb{Z})$ with the property:*

$$\sum_{k \in I} f_k(t) \geqslant c_1 \sqrt{t},$$

and moreover

$$|I| \leqslant c_2 t$$

for some positive constants c_1, c_2 not depending on t.

Proof. Denote

$$x_k(t) = tg(2\mu) = tg(\nu_k) = tg\left(2\frac{k}{t}\right), \quad \nu_k = 2\frac{k}{t}. \tag{7}$$

Then

$$f_k(t) = \sqrt{\frac{2}{\pi t \sqrt{1 - 4\mu^2}}} \cos\left(x_k(t) - \frac{\pi}{4}\right).$$

To prove the lemma we need to examine the points $x_k(t)$ modulo 2π. It is sufficient to prove that the number of points x_k in the interval $(0, \pi/2)$ has order t.

We use the equality

$$g(\nu_{k+1}) = g(\nu_k) + \frac{2}{t} g'(\nu_k) + \frac{4}{t^2} g''(\theta_k),$$

for some $\theta_k \in [\nu_k, \nu_{k+1}]$. Whence

$$x_{k+1}(t) = x_k(t) + 2g'(\nu_k) + \frac{4}{t} g''(\theta_k).$$

Thus, x_{k+1} and x_k differ on some angle, which for large t equals approximately $2g'(\nu_k)$. Let us find the derivative of g:

$$g'(\nu) = -\frac{\nu}{\sqrt{1 - \nu^2}} - \arccos(\nu) + \frac{\nu}{\sqrt{1 - \nu^2}} = -\arccos(\nu).$$

Thus for $\nu \in (0, 1)$, $g'(\nu)$ is negative and $g'(1) = 0$. Fix some small number ε. It is clear that there exist interval $(a, b) \subset (0, 1)$ and t_0 such that for all $x, y \in (a, b)$ and all $t > t_0$ the following inequality holds:

$$-2\varepsilon < 2g'(x) + \frac{4}{t} g''(y) < -\varepsilon.$$

Then for all $k \in [at/2, bt/2]$

$$-2\varepsilon < x_{k+1}(t) - x_k(t) < -\varepsilon. \tag{8}$$

Consider the set

$$I = \left\{ k \in \left[\frac{a}{2} t, \frac{b}{2} t\right] : \ x_k(t) \in \left(0, \frac{\pi}{2}\right) \mod 2\pi \right\}.$$

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By (8) we have:
$$c_1 t \leqslant |I| \leqslant c_2 t,$$
for some positive constants c_1, c_2 not depending on t. Then,
$$\sum_{k \in I} f_k(t) \geqslant c_1 \sqrt{\frac{2}{\pi t \sqrt{1 - b^2}}} \frac{1}{\sqrt{2}} t = c_3 \sqrt{t},$$
and the lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we come back to the proof of Theorem 3.1. Using Lemma 4 we put
$$q_k = \begin{cases} 1, & k \in I, \\ 0, & k \notin I. \end{cases}$$

Then by formula (6) and Lemma 3.6 we have
$$q_0\left(\frac{t}{2\omega_1}\right) = \sum_{k \in I} J_{2k}(t) = \frac{1}{2} \sum_{k \in I} f_k(t)\left(1 + O\left(\frac{1}{t}\right)\right) \geqslant c_1 \sqrt{t} + c_2 \frac{1}{\sqrt{t}}.$$

Theorem 3.1 is thus proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we prove part 2 of Theorem 2.1. For the sequence of time moments $T_1 < T_2 < \ldots < T_k \ldots$ denote by $q^1(t), q^2(t), \ldots, q^k(t), \ldots$ the solutions corresponding to those in the formulation of Theorem 3.1. We shall assume that $T_1 > T_0$ and
$$bT_k < aT_{k+1}$$
for all $k = 1, 2, \ldots$. This inequality guaranties that
$$\operatorname{supp}(q^i(0)) \cap \operatorname{supp}(q^j(0)) = \emptyset$$
for $i \neq j$, and we can define the sum:
$$q(0) = \sum_{i=1}^{\infty} q_i(0).$$

Due to linearity for the solution with the initial condition $q(0), p(0) = 0$ we have:
$$q(t) = \sum_{i=1}^{\infty} q_i(t).$$

To estimate $q(T_k)$ write
$$q(T_k) = \sum_{i=1}^{k-1} q^i(T_k) + q^k(T_k) + \sum_{i=k+1}^{\infty} q^i(T_k).$$

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By Proposition 2.1

$$|q_0^i(T_k)| \leqslant \sqrt{(b-a)T_i}.$$

Whence

$$\left| \sum_{i=1}^{k-1} q_0^i(T_k) \right| \leqslant \sqrt{b-a} \sum_{i=1}^{k-1} \sqrt{T_i}.$$

We estimate the third term above using Lemma 3.4 and get:

$$\left| \sum_{i=k+1}^{\infty} q_0^i(T_k) \right| \leqslant \left(e^{\alpha_k+1} \alpha_k \right)^{2M}, \quad \alpha_k = \omega_1 \frac{T_k}{M}$$

where $M = [aT_{k+1}] + 1$. Choose $T_{k+1} > T_k$ so that

$$e^{\alpha_k+1} \alpha_k < 1.$$

Then we have the estimate:

$$q_0(T_k) \geqslant c\sqrt{T_k} - \sqrt{b-a} \sum_{i=1}^{k-1} \sqrt{T_i} - 1.$$

The final condition for T_k is

$$\sqrt{b-a} \sum_{i=1}^{k-1} \sqrt{T_i} + 1 < \frac{c}{2} \sqrt{T_k}.$$

With such choice of the sequence T_k we have:

$$q_0(T_k) \geqslant \frac{c}{2} \sqrt{T_k}.$$

Thus the constructed sequence T_k provides initial condition and sequence t_k, satisfying the assertion of the second part of Theorem 2.1. Thus the theorem is proved.

4   Proof of Theorem 2.2

Plan of the proof is the following. First we will prove that finite dimensional distributions of $q_0(t+s)$ weakly converge as $t \to \infty$ to the finite dimensional distribution of some Gaussian stationary random process for $s \in [0, +\infty]$. This fact allows us to prove (2) in quite straightforward manner. Next without loss of generality we will suppose that $\omega_1 = 1/2$.

Define a family (parametrized by $t \geqslant 0$) of processes with smooth trajectories:

$$Q_t(s) = q_0(t+s), \ s \in [0, +\infty).$$

Define a process $X(s)$ as a series:

$$X(s) = \sum_{n \in \mathbb{Z}} \xi_n J_n(s), \tag{9}$$

where ξ_n are independent standard Gaussian random variables. Lemma 4.4 gives us:

$$\sum_{n \in \mathbb{Z}} J_n^2(s) = 1.$$

Therefore from Kolmogorov's two-series theorem follows the almost sure convergence of series in (9) for all $s \in \mathbb{R}$. Obviously $X(s)$ is a zero mean Gaussian random process. Let us calculate its covariance function again using lemma 4.4:

$$\mathrm{cov}(X(t), X(s)) = \sum_{n \in \mathbb{Z}} J_n(s) J_n(t) = J_0(t - s).$$

Therefore $X(s)$ is a stationary (in a wide sense) process with covariance function

$$C_X(s) = \mathrm{cov}(X(s), X(0)) = J_0(s).$$

Lemma 4.1 states that finite dimensional distribution of $Q_t(s)$ converges as $t \to \infty$ to the finite dimensional distribution of $(\sigma^2/2) X(s)$. Further on without loss of generality we will assume that $\gamma = \sigma^2/2 = 1$. The fact that maximum of $X(s)$ over $s \geqslant 0$ is infinite almost sure easily follows from the classical theory of stationary Gaussian process (see [8]). Therefore intuitively it is clear that maximum of $q_0(t)$ is infinite with probability one. But we can not use this arguments in strong way while we will not prove a weak convergence of the corresponding processes. We will not follow this way. Instead of proving weak convergence we derive (2) directly from Lemma 4.1.

We have the following equalities:

$$P\left\{\sup_{t \geqslant 0} q_0(t) = +\infty\right\} = P\left\{\bigcap_{a=1}^{+\infty} \{\sup_{t \geqslant 0} q_0(t) \geqslant a\}\right\} = \lim_{a \to +\infty} P\left\{\sup_{t \geqslant 0} q_0(t) \geqslant a\right\}.$$

Now we prove that

$$P\left\{\sup_{t \geqslant 0} q_0(t) \geqslant a\right\} = 1 \tag{10}$$

for all a.

Fix an arbitrary $\varepsilon > 0$. Note that due to lemma 4.3 for all $\varepsilon' > 0$ and all $N \geqslant 1$ there exist $s_1, \ldots_N$ such that

$$|\mathrm{cov}(X(s_i), X(s_j))| = |C(s_i - s_j)| \leqslant \varepsilon' \tag{11}$$

for all $i \neq j$. In other words, $X(s_1), \ldots, X(s_N)$ are "almost" independent. Indeed, put $s_k = k\delta$ where $\delta^{-1/3} < \varepsilon'$. Then using lemma 4.3 we obtain:

$$|\mathrm{cov}(X(s_i), X(s_j))| = |J_0(s_i - s_j)| \leqslant |i - j|^{-1/3} \delta^{-1/3} \leqslant \varepsilon'.$$

Further we will choose ε' and N explicitly and they will depend on ε. Now let $s_1, \ldots, s_N$ satisfy (11). For all $T \geqslant 0$ we have the bound

$$P\left\{\sup_{t \geqslant 0} q_0(t) \geqslant a\right\} \geqslant P\left\{\sup_{k=1,\ldots,N} q_0(T + s_k) \geqslant a\right\}. \tag{12}$$

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From lemma 4.1 it follows that there is $T_0 \geqslant 0$ such that for all $T \geqslant T_0$ the following inequality holds:

$$\left| P\left\{ \sup_{k=1,\ldots,N} q_0(T + s_k) \geqslant a \right\} - P\left\{ \sup_{k=1,\ldots,N} X(s_k) \geqslant a \right\} \right| \leqslant \varepsilon. \tag{13}$$

Now we want to use lemma 4.2 to estimate $P\left\{\sup_{k=1,\ldots,N} X(s_k) \geqslant a\right\}$. Note that since $p(0) = \Phi(a) < 1$ and $p(\delta)$ is an increasing continuous function ($p(\delta)$ is defined in (18)), there exist small δ' and a number $q < 1$ such that for all $\delta < \delta'$ the following inequality holds: $p(\delta) < q$. Now suppose that $N\varepsilon' < \delta'$. Lemma 4.2 gives us

$$P\left\{ \sup_{k=1,\ldots,N} X(s_k) \geqslant a \right\} \geqslant 1 - q^N$$

which is greater than $1 - \varepsilon$ for sufficiently large N. Using (12) and (13) we obtain

$$P\left\{ \sup_{t \geqslant 0} q_0(t) \geqslant a \right\} \geqslant 1 - 2\varepsilon.$$

Since ε is arbitrary, we have proved (10). The proof for the inf is the same. This completes the proof of the theorem.

Lemma 4.1. *For all $s_1, \ldots, s_m \geqslant 0$ the following convergence of distributions holds:*

$$\mathrm{Law}(Q_t(s_1), \ldots, Q_t(s_m)) \to \mathrm{Law}(\gamma X(s_1), \ldots, \gamma X(s_m)), \quad \gamma = \frac{\sigma^2}{2}$$

as $t \to \infty$.

The proof is straightforward and based on the continuity theorem for characteristic function. From formula (5) we have:

$$q_0(t) = \sum_{n \in \mathbb{Z}} q_n(0) J_{2n}(t).$$

Consider the characteristic function of the random vector $Q_t(s_1), \ldots, Q_t(s_m)$:

$$f_t(u_1, \ldots, u_m) = E \exp\left(i \sum_{j=1}^{m} u_j Q_t(s_j) \right).$$

Due to dominated convergence theorem and independence of $q_k(0)$ we obtain:

$$f_t(u_1, \ldots, u_m) = E \exp\left(i \sum_{n \in \mathbb{Z}} q_n(0) \sum_{j=1}^{m} u_j J_{2n}(t + s_j) \right) = \prod_{n \in \mathbb{Z}} h(\phi_n(t, \bar{u})),$$

where $h(u) = E \exp(i q_0(0) u)$ is the characteristic function of $q_0(0)$, and

$$\phi_n(t, \bar{u}) = \sum_{j=1}^{m} u_j J_{2n}(t + s_j),$$

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with $\bar{u} = (u_1, \ldots, u_m)$. Fix $u_1, \ldots, u_m$. Because of lemma 4.3 for all $\varepsilon > 0$ we can choose $t_0 \geqslant 0$ such that for all $t \geqslant t_0$ all points $\phi_n(t, \bar{u})$, $n \in \mathbb{Z}$ lie in ε-neighborhood of zero. And since $h(0) = 1$, we can consider the principal branch of logarithm:

$$f_t(u_1, \ldots, u_m) = \exp\Big(\sum_{n \in \mathbb{Z}} \mathrm{Log}(h(\phi_n(t, \bar{u})))\Big). \tag{14}$$

From smoothness it follows that for sufficiently small u the following formula holds

$$\mathrm{Log}(h(u)) = -\frac{1}{2}\sigma^2 u^2 + O(u^3).$$

Hence we have

$$\mathrm{Log}(h(\phi_n(t, \bar{u}))) = -\frac{1}{2}\sigma^2 \phi_n^2(t, \bar{u}) + O(\phi_n^3(t, \bar{u})). \tag{15}$$

Using lemma 4.4 we get

$$\sum_{n \in \mathbb{Z}} \phi_n^2(t, \bar{u}) = \sum_{j,k=1,\ldots,m} u_j u_k \sum_{n \in \mathbb{Z}} J_{2n}(t + s_j) J_{2n}(t + s_k)$$

$$= \frac{1}{2} \sum_{j,k=1,\ldots,m} u_j u_k \left(J_0(2t + s_j + s_k) + J_0(s_j - s_k) \right)$$

$$\longrightarrow \frac{1}{2} \sum_{j,k=1,\ldots,m} u_j u_k J_0(s_j - s_k)$$

as $t \to \infty$. Next we study the sum of cubes. Note that due to Hölder inequality we have an estimate:

$$|\phi_n^3(t, \bar{u})| \leqslant m^2 \sum_{j=1}^m u_j^3 J_{2n}^3(t + s_j).$$

From this bound and lemma 4.3 we obtain:

$$\left| \sum_{n \in \mathbb{Z}} \phi_n^3(t, \bar{u}) \right| \leqslant m^2 \sum_{j=1}^m u_j^3 \sum_{n \in \mathbb{Z}} |J_{2n}^3(t + s_j)| \leqslant m^2 t^{-1/3} \sum_{j=1}^m u_j^3 \sum_{n \in \mathbb{Z}} J_{2n}^2(t + s_j)$$

$$= m^2 t^{-1/3} \sum_{j=1}^m u_j^3 \frac{J_0(2(t + s_j)) + J_0(0)}{2} \longrightarrow 0 \tag{16}$$

as $t \to \infty$. In equality (16) we applied lemma 4.4.

Thus formulas (14) and (15) give us:

$$\lim_{t \to \infty} f(t, u_1, \ldots, u_m) = \exp\Big(-\frac{\sigma^2}{4} \sum_{j,k=1,\ldots,m} u_j u_k J_0(s_j - s_k)\Big).$$

Applying the continuity theorem for characteristic functions we obtain the assertion of lemma 4.1. This completes the proof of the lemma.

Lemma 4.2. *Let* $X_1, \ldots, X_N$ *be a zero mean Gaussian random vector with* $N \geqslant 2$ *such that:*

$$DX_k = 1, \quad |\text{cov}(X_i, X_j)| \leqslant \varepsilon',$$

for all $k = 1, \ldots, N$ *and all* $i \neq j$ *and some* $\varepsilon' > 0$. *Assume the following inequality holds:*

$$N\varepsilon' \leqslant \delta < \frac{1}{2}. \tag{17}$$

Then there is the estimate:

$$P\left\{\sup_{k=1,\ldots,N} X_k \geqslant a\right\} \geqslant 1 - p^N, \quad p = p(\delta) = \sqrt{\frac{1+\delta}{1-\delta}} \, \Phi(a\sqrt{1+\delta}) \tag{18}$$

where $\Phi(x)$ *is a cumulative distribution function of standard normal distribution:*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

We have:

$$P\left\{\sup_{k=1,\ldots,N} X_k \geqslant a\right\} = 1 - P\left\{\sup_{k=1,\ldots,N} X_k < a\right\}.$$

Denote by C the covariance matrix of the random vector X,

$$C = (\text{cov}(X_i, X_j))_{i,j=1,\ldots,N}.$$

Due to Gershgorin circle theorem all eigenvalues of C lye in the circle with radius $(N-1)\varepsilon' < 1/2$ and center at 1. Therefore C is invertible and we can write

$$P\left\{\sup_{k=1,\ldots,N} X_k < a\right\} = \frac{1}{(\sqrt{2\pi})^N \sqrt{\det(C)}} \int_{[-\infty,a]^N} \exp\left\{-\frac{(x, C^{-1}x)}{2}\right\} dx,$$

where by $(,)$ we denoted the standard Euclidean scalar product in $\mathbb{R}^N$. We have an obvious inequality:

$$(x, C^{-1}x) \geqslant \frac{1}{\lambda}|x|^2$$

where $|x|$ is standard Euclidean norm of the vector x in $\mathbb{R}^N$ and λ is the maximal eigenvalue of C. Hence we obtain for the probability

$$P\left\{\sup_{k=1,\ldots,N} X_k < a\right\} \leqslant \frac{1}{(\sqrt{2\pi})^N \sqrt{\det(C)}} \int_{[-\infty,a]^N} \exp\left\{-\frac{|x|^2}{2\lambda}\right\} dx$$

$$= \frac{\sqrt{\lambda}^N}{\sqrt{\det(C)}} \Phi^N(\sqrt{\lambda}a). \tag{19}$$

Evidently we have $\det(C) = \lambda_1 \ldots \lambda_N$ where $\lambda_1, \ldots, \lambda_N$ are eigenvalues of C. Thus applying Gershgorin circle theorem we get:

$$\det(C) \geqslant (1-\delta)^N, \quad \lambda \leqslant 1 + \delta.$$

Putting these inequalities to (19) we obtain (18). This completes the proof of the lemma.

Lemma 4.3. *For all $n \in \mathbb{Z}$ and every $t \in \mathbb{R}$ the following inequality holds*

$$|J_n(t)| \leqslant \min\{|n|^{-1/3}, |t|^{-1/3}\}.$$

For the proof see [5, 6].

Lemma 4.4 (Neumann's additional theorem). *For all $\varphi, t_1, t_2 \in \mathbb{R}$ the following formula holds :*

$$J_0(\bar{t}) = \sum_{n \in \mathbb{Z}} J_n(t_1) J_n(t_2) \cos(n\varphi), \tag{20}$$

where $\bar{t} = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos \varphi}$. As a particular case of (20) we have:

$$\sum_{n \in \mathbb{Z}} J_{2n}(t_1) J_{2n}(t_2) = \frac{1}{2} \left(J_0(t_1 + t_2) + J_0(t_1 - t_2) \right). \tag{21}$$

One can find Neumann's additional theorem (20) in classical book [7], p. 358–359. Equality (21) immediately follows from (20) if we put $\varphi = 0$ and then $\varphi = \pi$ and next sum up two expressions. Here we used that $J_0(t) = J_0(-t)$.

Let us make a remark about weak convergence of the process $q_0(t+s)$. Due to additional theorem [7, p. 30], we have :

$$J_{2k}(t + s) = \sum_{n \in \mathbb{Z}} J_n(s) J_{2k-n}(t).$$

Therefore due to formula (5) we obtain

$$q_0(t + s) = \sum_{n \in \mathbb{Z}} J_n(s) y_n(t),$$

where

$$y_n(t) = \sum_{k \in \mathbb{Z}} q_k(0) J_{2k-n}(t).$$

It is not hard to prove that $y_n(t)$ converges to ξ_n in distribution as $t \to \infty$, where ξ_n are standard normal independent random variables. At the next step one can prove the relative compactness of the family $q_0(t + s), s \in [0, S]$ for some $S \geqslant 0$, parametrized by t. We have no need for weak convergence so we have omitted this proof.

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