ISSN 0001-4346, Mathematical Notes, 2012, Vol. 92, No. 2, pp. 237–248.

Analytic Dynamics of a One-Dimensional System of Particles with Strong Interaction

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Received July 15, 2011; in final form, December 8, 2011

Abstract—We consider the dynamics of a system of N particles on the circle with interaction of nearest neighbors, a Coulomb potential, and an analytic external force. The trajectories are real analytic functions of time. However, the series for them converge only for sufficiently small times. For zero initial velocities and a uniform initial location of particles, we prove N-dependent estimates on the coefficients of this series.

DOI: 10.1134/S0001434612070255

Keywords: *system of particles with strong interaction, Coulomb potential, electric current, analytic external force.*

1. INTRODUCTION

1.1. Statement of the Problem and Main Results

Consider a system of N point particles $i = 1, 2, \dots, N$ on a half-open interval $[0, L) \subset \mathbb{R}$ with periodic boundary conditions, i.e., on the circle S_L of length L. Initially, they are located at the points

$$0 = x_1(0) < \cdots < x_N(0) < L.$$

The trajectories $x_i(t)$ are determined from the system of N equations

$$\frac{d^2 x_i}{dt^2} = -\frac{\partial U}{\partial x_i} + F(x_i); \tag{1}$$

the interaction between the particles is of the form

$$U(\{x_i\}) = \sum_{\langle i,i-1\rangle} V(x_i - x_{i-1}),$$

where the sum is taken over all pairs of neighbors on the circle. We assume that the potential $V(x) = V(-x) = 1/r$, $r = |x|$, is Coulomb and set

$$f(r) = -\frac{dV(r)}{dr} = r^{-2}.$$

Note that, because of strong repulsion at close distances, the particles in motion do not change their order. Let $F(x)$ denote an external force.

Fixed points of such systems were studied in [1], [2]. Questions pertaining to dynamics are significantly more complicated. Certainly, the solution of system (1) exists and is unique (under arbitrary initial conditions) on the whole time interval; however, it is rather hard to obtain more detailed information about the trajectories of the particles (if N is sufficiently large) without developing a special technique. If F is analytic, then, as is well known [3], the solution can be expressed as a power series expansion in t in some neighborhood of the point $t = 0$.

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We consider the natural initial conditions: for all i,

$$\Delta = \Delta_i(0) = x_{i+1}(0) - x_i(0) = \frac{L}{N}, \qquad v_i(0) = 0, \tag{2}$$

and it is convenient to assume that $x_1(0) = 0$. Note that this configuration is a fixed point in the case of a zero external force. In this paper, we obtain estimates of the radius of convergence for such initial conditions. We search for a solution of the form

$$v_i(t) = \sum_{j=1}^{\infty} c_{i,j} t^j. \tag{3}$$

Theorem 1. *Let F be analytic on the circle S_L. Then*

1) *for all $j = 1, 2, \dots$, there exist numbers $b_j < \infty$ not depending on N such that, for all i and j and for all N,*

$$|c_{ij}| < b_j N^{(j-1)/2};$$

2) *let, in addition, for some $C_F > 0$ and all x and k,*

$$|F^{(k)}(x)| \le C_F^{k+1}.$$

Then there exists a constant $0 < \chi < \infty$ not depending on N such that, for all i, j, the following estimate holds:

$$|c_{ij}| < \chi^j N^{5j/6-3/2}.$$

This implies that the radius of convergence $R = R(N)$ of the series (3) has the lower bound $R > \chi^{-1} N^{-5/6}$. From the proof of the first assertion of the theorem in Sec. 2.2, we can conclude that an upper bound for the radius of convergence can be of order $1/\sqrt{N}$, but this has not been proved yet. In the proof of the second assertion of the theorem, we give an explicit estimate for χ. Also we present explicit formulas for c_{ij} for $j = 1, 2, 3, 4$.

The goal and essence of the present paper is best illustrated by its physical motivation.

1.2. On the Riddle of the Electric Current

Mathematical questions of statistical physics have been extensively developed for equilibrium systems on the lattice; as to continuous space, it appears that they have been sufficiently developed only for gases with small inverse temperature or density. For many other cases, there are even no mathematical statements of problems. One of such cases is the direct electric current. Considered on the macrolevel, such a current is described adequately by Ohm's law, while, on the microlevel, it is considered in all textbooks on solid-state physics as a system of free (or weakly dependent) electrons each of which is accelerated by an external force and retarded by the external medium: both physicists and mathematicians studied one-particle models with constant accelerating external force and various versions of the external medium absorbing the energy of particles. (There were at least twenty such versions, the first of which being the Drude model of 1900.)

Nevertheless, there is still an important question unanswered: Where does the accelerating force come from? The fact is that, in any electric power line, the force acts only for a distance of several meters from the generator, the turbine, etc. Here is what R. Feynman writes in this connection in his famous "Lectures in Physics" (Vol 2, Sec. 16-2):

"... The force pushes the electrons along the wire. But why does this move the galvanometer, which is so far from the force? Because when the electrons which feel the magnetic force try to move, they push—by electric repulsion—the electrons a little farther down the wire; they, in turn, repel the electrons a little farther on, and so on for a long distance. An amazing thing. It was so amazing to Gauss and Weber—who first built a galvanometer—that they tried to see how far the forces in the wire would go. They strung the wire all the way across the city ... "

So writes the famous physicist. We must state that this problem is, in general, totally ignored in other books and papers.

Hence we already see that the electrons pushing each other constitute a strongly interacting system of particles and such an interaction can only be Coulomb repulsion. For a mathematician, it is natural to study this phenomenon by starting first with the simplest model. Such a model will be considered in what follows.

In fact, there is not just one problem, but many. For example, we must explain why the steady-state current velocity is constant and very small (0.1–10 mm/s), while the current is set up almost instantaneously. The main idea behind the fact that the steady-state velocity is constant and small was put forward in a somewhat different model in [4], but that model did not encompass Coulomb systems.

In this paper, we attempt to study the second problem. If the coefficients c_k in the time series

$$v = \sum_{k=1}^{\infty} c_k t^k$$

for the particle velocity increase as N^{ak}, $a > 0$, where N is the number of particles, then it is natural to expect that the velocity of order 1 is established before the instant of time of order $t = N^{-a}$, which is "almost instantaneous" from the physical point of view. We were not able to find a rigorous proof of the fact that the velocity setup time is small. however, the estimates of the coefficients given in this paper make this fact very likely.

Since such estimates are sufficiently complicated even for the simplest model (see also [1], [2]), such problems were not studied by physicists.

Numerous papers, beginning with Bogolyubov's papers, dealing with the dynamics of multiparticle systems, had a completely different orientation. In most cases, the existence of a thermodynamic limit of the dynamics was proved. This means that, for a certain period of time, the given arbitrary particle is significantly affected by only a bounded number of (adjacent) particles, and this remains valid in an infinite volume. One of the main techniques used here is also the expansion (for example, of the velocity) in a time series whose coefficients, however, are bounded uniformly with respect to the number of particles and the velocity is small for small times under zero initial conditions; see [5]–[12]. In our case, the thermodynamic passage to the limit is meaningless, because our problem is more intricate.

2. PROOF

2.1. Equations for the Coefficients

Let us fix the initial data $x_i(0)$, $v_i(0)$ just as in (2) and consider the trajectories $x_i(t) \in S$ on the interval $0 \le t < t_0$ for some instant of time $t_0 = t_0(N) > 0$. Setting

$$\Delta_i(t) = x_{i+1}(t) - x_i(t), \qquad \Delta = \Delta_i(0) = \frac{L}{N},$$

we obtain the equations

$$\frac{dv_i}{dt} = f(x_i(t) - x_{i-1}(t)) - f(x_{i+1}(t) - x_i(t)) + F(x_i(t))$$

or

$$\frac{dv_i}{dt} = f\biggl(\Delta + \int_0^t [v_i(t_1) - v_{i-1}(t_1)]\,dt_1\biggr) - f\biggl(\Delta + \int_0^t [v_{i+1}(t_1) - v_i(t_1)]\biggr)\,dt_1\biggr) + F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr).$$

Integral equations. The equivalent system of integral equations

$$v_i(t) = \int_0^t \biggl[f\biggl(\Delta + \int_0^t \bigl[v_i(t_1) - v_{i-1}(t_1)\bigr]\,dt_1\biggr) \\ - f\biggl(\Delta + \int_0^t [v_{i+1}(t_1) - v_i(t_1)]\,dt_1\biggr) + F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr)\biggr]\,dt \tag{4}$$

can be rewritten as

$$v_i(t) = \int_0^t \biggl((\Delta + R_{i-1}(t))^{-2} - (\Delta + R_i(t))^{-2} + F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr)\biggr)\,dt, \tag{5}$$

where

$$R_{i-1}(t) = \int_0^t (v_i(t_1) - v_{i-1}(t_1))\,dt_1.$$

In what follows, we shall need some notation related to discrete derivatives. Let a function $g(i)$ be given on the interval $[0, N] \subset \mathbb{Z}$ with periodic boundary conditions (i.e., a periodic function on $\mathbb{Z}$ with period N). Let us call

$$(\nabla^+ g)(i) = g(i+1) - g(i) \qquad \text{and} \qquad (\nabla^- g)(i) = g(i) - g(i-1) \tag{6}$$

its *right* and *left derivative*, respectively. Note that they commute and the following relation holds:

$$\nabla^+(gf)(i) = f(i+1)(\nabla^+ g)(i) + g(i)(\nabla^+ f)(i) = (Sf)(\nabla^+ g) + g(\nabla^+ f), \tag{7}$$

where S is the shift operator:

$$(Sf)(i) = f(i+1).$$

In what follows, the discrete differentiation operators will act on the indices i. If the function $f(i)$ is independent of i, then its differentiation yields zero.

Let us return to the main equations and rewrite them as follows:

$$v_i(t) = \int_0^t dt\,\biggl[(-\nabla^-((\Delta + R_i(t))^{-2}) + F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr)\biggr].$$

The following representation of the integrand will be useful:

$$\begin{aligned}
&(\Delta + R_{i-1}(t))^{-2} - (\Delta + R_i(t))^{-2} + F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr) \\
&\quad = \Delta^{-2}\biggl(1 + \frac{R_{i-1}}{\Delta}\biggr)^{-2} - \Delta^{-2}\biggl(1 + \frac{R_i}{\Delta}\biggr)^{-2} + F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr) \\
&\quad = F(x_i(0)) + \sum_{m=1}^{\infty} d_m[\Delta^{-2-m}(R_{i-1}^m - R_i^m)] + \biggl[F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr) - F(x_i(0)\biggr] \\
&\quad = F(x_i(0)) + \sum_{m=1}^{\infty} d_m \Delta^{-2-m}(R_{i-1}^m - R_i^m) + \biggl[F\biggl(x_i(0) + \int_0^t v_i(t_1)\,dt_1\biggr) - F(x_i(0))\biggr],
\end{aligned}$$

where

$$d_m = (-1)^m(m+1).$$

If F is analytic on S_L, then there exists a sufficiently small $\epsilon > 0$ such that, for any $x_0 \in S_L$ and for all $x \in [x_0 - \epsilon, x_0 + \epsilon)$, the following expansion in a convergent series is valid:

$$F(x) = F(x_0) + \sum_{k=1}^{\infty} \frac{F^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Finally, we obtain

$$v_i(t) = F(x_i(0))t + \int_0^t \sum_{m=1}^{\infty} d_m \Delta^{-2-m}[-\nabla^- R_i^m)]\,dt + \sum_{k=1}^{\infty} \int_0^t \frac{F^{(k)}(x_i(0))(\int_0^t v_i(t_1)\,dt_1)^k}{k!}\,dt. \tag{8}$$

Recurrence equations. Using (3) and

$$R_{i-1}(t) = \sum_{j=1}^{\infty} (c_{i,j} - c_{i-1,j}) \frac{t^{j+1}}{j+1}, \tag{9}$$

$$R_i - R_{i-1} = \sum_{j=1}^{\infty} (c_{i+1,j} - 2c_{i,j} + c_{i-1,j}) \frac{t^{j+1}}{j+1} \tag{10}$$

and substituting the series (3) into (8), we see that the right-hand side of (8) is also a power series expansion in t with well defined coefficients.

Let us search for $c_{i,j}$ by equating of the coefficients of t^j. For $j = 1, 2$, the equations immediately give the explicit expressions

$$c_{i1} = F(x_i(0)), \qquad c_{i,2} = 0, \tag{11}$$

because the other summands on the right-hand side of (8) are of greater order in t. For $j \geq 3$, the equations for the coefficients of t^j are of the form

$$c_{ij} = \frac{1}{j}\biggl[\sum_{m=1}^{\infty} d_m \Delta^{-2-m}(-\nabla^- R_i^m) + \sum_{k=1}^{\infty} \frac{F^{(k)}(x_i(0))(\int_0^t v_i(t_1)\,dt_1)^k}{k!}\biggr]_{j-1}, \tag{12}$$

where, for the power series

$$\phi(t) = \sum_{k=0}^{\infty} a_k t^k,$$

we put $[\phi(t)]_j = a_j$. For $j > 2$, the coefficients $c_{i,j}$ are found from a recurrence relation; note that the c_{ij} depend only on $c_{i,k}$ with $k \leq j-2$. Indeed, on the right-hand side of the equation for $c_{i,j}$, there cannot be $c_{i,k}$ with $k \geq j-1$, because, in view of (9), each of the c_{ik} appears together with t^{k+1}.

In that case, the main equations take the form

$$\begin{aligned} c_{ij} = \frac{1}{j} \sum_{m=1}^{\infty} d_m \Delta^{-2-m}\biggl(-\nabla^-\biggl[\biggl(\sum_{j=1}^{\infty}(c_{i+1,j} - c_{i,j})\frac{t^{j+1}}{j+1}\biggr)^m\biggr]_{j-1}\biggr) \\ + \sum_{k=1}^{\infty} \frac{F^{(k)}(x_i(0))}{k!}\biggl[\biggl(\sum_{j=1}^{\infty} c_{i,j}\frac{t^{j+1}}{j+1}\biggr)^k\biggr]_{j-1}. \end{aligned} \tag{13}$$

We have

$$\biggl[\biggl(\sum_{j=1}^{\infty} c_{i,i}\frac{t^{j+1}}{j+1}\biggr)^k\biggr]_{j-1} = \sum_{j_1+\dots+j_m=j-m-1} \frac{c_{i,j_1}}{j_1+1} \cdots \frac{c_{i,j_k}}{j_k+1}, \tag{14}$$

where $\sum_{j_1+\dots+j_m=j-m-1}$ is the sum over all ordered collections $j_1, \dots, j_k$, (some of which can be identical) such that

$$(j_1+1) + \dots + (j_k+1) = k + j_1 + \dots + j_k = j-1, \tag{15}$$

whence

$$k \leq j_1 + \dots + j_k = j-1-k \leq j-2, \qquad k \leq \biggl[\frac{j-1}{2}\biggr]. \tag{16}$$

Similarly,

$$\Big[\Big(\sum_{j=1}^{\infty}(c_{i+1,j}-c_{i,j})\frac{t^{j+1}}{j+1}\Big)^m\Big]_{j-1}=\sum_{j_1,\dots,j_m}^{(j-1,m)}\frac{\nabla^+c_{i,j_1}}{j_1+1}\cdots\frac{\nabla^+c_{i,j_m}}{j_m+1};$$

here (15) and (16) hold with k replaced by m. Therefore, the equation can be written as

$$c=Gc+c^{(0)}, \tag{17}$$

where c is the vector $c=\{c_{ij}\}$, the constant term $c^{(0)}=\{c_{ij}^{(0)}\}$ is

$$c_{i1}^{(0)}=F(x_i(0)),\qquad c_{ij}^{(0)}=0,\quad j\ge 2, \tag{18}$$

and the nonlinear operator G is of the form

$$\begin{aligned} c_{i1}&=c_{i1}^{(0)}=F(x_i(0)),\qquad c_{i2}=0,\\ c_{ij}&=(Gc)_{ij}=-\sum_{m=1}^{[(j-1)/2]}\sum_{j_1+\dots+j_m=j-m-1}A_{ij}(m;j_1,\dots,j_m)\\ &\quad+\sum_{k=1}^{[(j-1)/2]}\sum_{j_1+\dots+j_k=j-k-1}B_{ij}(k;j_1,\dots,j_k)\end{aligned} \tag{19}$$

for $j\ge 3$, where

$$A_{ij}(m;j_1,\cdots,j_m)=\frac{1}{j}\,d_m\Delta^{-2-m}\nabla^-\Big(\frac{\nabla^+c_{i,j_1}}{j_1+1}\cdots\frac{\nabla^+c_{i,j_m}}{j_m+1}\Big), \tag{20}$$

$$B_{ij}(k;j_1,\dots,j_k)=\frac{1}{j}\frac{1}{k!}F^{(k)}(x_i(0))\frac{c_{i,j_1}}{j_1+1}\cdots\frac{c_{i,j_k}}{j_k+1}. \tag{21}$$

Further, let $F_{i,k,q}$ denote any discrete derivative of the form

$$\Big(\prod_{p=1}^{q}\nabla^{s(p)}\Big)F^{(k)}(x_i(0)),$$

where $s(p)=\pm$. For estimates, the choice of $s(p)$ is not important at all. Denote $F_{i,k}=F_{i,k,0}$.

Explicit expressions for c_{i3} and c_{i4} are readily obtained if, in Eqs. (19), we take into account only terms with $k=1$ and $m=1$, because $k,m\le[(j-1)/2]\le 1$. We have

$$\begin{aligned} c_{i3}&=-\frac{1}{3}d_1\Delta^{-3}\nabla^-\nabla^+\frac{c_{i1}}{2}+\frac{1}{3}F^{(1)}(x_i)\frac{c_{i1}}{2}=\frac{1}{6}(d_1\Delta^{-3}F_{i,0,2}+F_{i,0,0}F_{i,1,0}),\\ c_{i4}&=-\frac{1}{4}d_1\Delta^{-3}\nabla^-\Big(\nabla^+\frac{c_{i1}}{2}\Big)^2+\frac{1}{4}F^{(1)}(x_i)\frac{c_{i1}^2}{4}=\frac{1}{8}\Big(-d_1\Delta^{-3}F_{i,0,2}F_{i,0;1}+\frac{1}{2}F_{i,1,0}F_{i,0,0}^2\Big).\end{aligned}$$

The formulas

$$\begin{aligned} F_{i,0,1}&=F_{i+1,0,0}-F_{i,0,0}=\int_{x_i}^{x_{i+1}}F^{(1)}(x)\,dx, &&|F_{i,0,1}|\le C_F^2\Delta,\\ F_{i,0,2}&=(F_{i+2,0,0}-F_{i+1,0,0})-(F_{i+1,0,0}-F_{i,0,0})\\ &=\int_{x_i}^{x_{i+1}}\Big(\int_x^{x+\Delta}F^{(2)}(y)\,dy\Big)\,dx, &&|F_{i,0,2}|\le C_F^3\Delta^2,\end{aligned} \tag{22}$$

imply that

$$|c_{i3}|\le\frac{1}{3}C_F^3\Big(\Delta^{-1}+\frac{1}{2}\Big),\qquad |c_{i4}|\le\frac{1}{4}C_F^5+\frac{1}{16}C_F^4.$$

2.2. Estimate of the Principal Exponent

It follows from the recurrence formulas (20) and (21) that the coefficients c_{ij} are finite and depend on i, j, and N. First, we consider them as functions of N for fixed i, j. In other words, we prove the first part of the theorem. Let us introduce the notion of principal exponent

$$I(\xi) = \limsup_{N\to\infty} \frac{\ln|\xi|}{\ln N}$$

for the quantity ξ depending on N. In simple terms, this notion indicates that the principal order of the asymptotics ξ is $N^{I(\xi)}$.

We shall consider the algebra $\mathbf{A}$ of polynomials in a countable number of (commuting) variables $F_{i,k,q}$, $i = 1, \dots, N$, $k, q = 0, 1, 2, \dots$ with real coefficients independent of F. For any monomial M from this algebra, we denote

$$Q(M) = -\sum q$$

over all q in this monomial. The natural mapping of the algebra $\mathbf{A}$ onto the subalgebra $\mathbf{A}_0$ generated by all the $F_{i,k} = F_{i,k,0}$ is defined by the successive substitutions

$$F_{i,k,q} = F_{i+1,k,q-1} - F_{i,k,q-1}$$

or

$$(\nabla^+)^n = (S-1)^n = \sum_{k=0}^{n} C_n^k (-1)^k S^{n-k}.$$

Lemma 1. *For any monomial $M \in \mathbf{A}$,*

$$I(M) \le Q(M).$$

Proof. In order to prove the lemma, it suffices to show that

$$I(F_{i,q}) \le Q(F_{i,q}) = -q.$$

Just as in (22), to do this, we use induction on q.

For any polynomial $P = \sum a_r M_r$ with (different) monomials M_r and coefficients a_r not depending on F, but, possibly, depending on N, we define

$$Q(P) = \max_r (I(a_r) + Q(M_r)),$$

which is consistent with the previous definition. Then, for any polynomial P, we have

$$I(P) \le \max_r (I(a_r) + I(M_r)) \le \max_r (I(a_r) + Q(M_r)).$$

Note that, for every pair of polynomials P_1 and P_2, the following inequality holds:

$$Q(P_1 P_2) \le Q(P_1) + Q(P_2).$$

We also have

$$Q(\nabla^+ P) \le Q(P) - 1, \qquad Q(\nabla^- \nabla^+ P) \le Q(P) - 2. \tag{23}$$

By the *degree* $\deg P$ *of the polynomial* $P = \sum a_r M_r$ we shall mean the greatest degree of its monomials. □

Lemma 2. *For $j > 1$, c_{ij} is a polynomial in the algebra $\mathbf{A}_0$ of degree at most $j-1$.*

Proof. We have already noticed this fact for $j = 1, 2, 3, 4$. Note that

$$\deg(\nabla^\pm P) = \deg P.$$

Further, we can use induction: in formula (20), the degree will be $j-2$, while, in (21), the degree will be $j-1$. □

Note that the recurrence formulas define c_{ij} for all functions $F(x)$, not necessarily analytic. Therefore, the following statement is meaningful.

Lemma 3. *Let F be infinitely differentiable. Then, for all i, j,*

$$I(c_{ij}) \le Q(c_{ij}) \le \frac{j-1}{2}.$$

Proof. Since

$$Q(c_{ij}) = 0, \quad j = 1, 2, \qquad Q(c_{i,3}) = 1, \qquad Q(c_{i,4}) = 0,$$

the statement is valid for $j = 1, 2, 3, 4$. Let us prove the lemma by induction on j. Suppose that

$$Q(c_{ij}) \le \frac{j-1}{2}$$

for all $j = 1, 2, \dots, J-2$.

Then, for given $m, j_1, \dots, j_m$, using (15) and (16), we obtain

$$\begin{aligned} Q(A_{iJ}(m; j_1, \dots, j_m)) &\le 2 + m - 1 + Q(c_{ij_1}) + \dots + Q(c_{ij_m}) - m \\ &\le 1 + \frac{1}{2}(j_1 + \dots + j_m) - \frac{m}{2} = 1 + \frac{1}{2}(J - m - 1) - \frac{m}{2}, \end{aligned}$$

because, in view of (23), (-1) and $(-m)$ are added from the action of the discrete derivative operators applied to the corresponding monomials. The maximum of the last expression is attained at $m = 1$. Hence

$$Q(A_{iJ}(m; j_1, \dots, j_m)) \le \frac{J-1}{2}.$$

Similarly, for $B_{iJ}(k : j_1, \dots, j_k)$, the following inequalities hold:

$$Q(B_{iJ}(k; j_1, \dots, j_k)) \le \frac{1}{2}(J - 1 - k) - \frac{k}{2} < \frac{J-1}{2}.$$

This yields $Q(c_{iJ}) \le (J-1)/2$, and hence $I(c_{iJ}) \le Q(c_{iJ}) \le (J-1)/2$. □

2.3. The Radius of Convergence

Here we prove the second assertion of Theorem 1. In the proof, it is convenient to write N instead of N/L and assume $C_F \ge 1$.

We shall use the majorization principle for infinite systems of recurrence equations and inequalities: for example if we are given two systems of equations

$$c_{ij}^{(q)} = P^{(q)}(c_{i1}^{(q)}, \dots, c_{i,j-2}^{(q)}), \qquad q = 1, 2,$$

where the $P^{(q)}$ are polynomials with coefficients $p_\alpha^{(q)}$ such that $p_\alpha^{(2)} \ge 0$, $|p_\alpha^{(1)}| \le p_\alpha^{(2)}$ for all α, and $|c_{ij}^{(1)}| \le c_{ij}^{(2)}$ for $j = 1, 2, 3, 4$, then $|c_{ij}^{(1)}| \le c_{ij}^{(2)}$ for all the j. We shall presently introduce one of such systems obtained from the one-particle problem (i.e., the problem with $N = 1$) with a specially chosen external force. Another auxiliary system $\beta(c_{ij})$ with positive coefficients will be introduced later.

The one-particle problem. For $j = 1, 2, \dots$ and a fixed a, let us set Then the following statement is valid.

Lemma 4. *For $j = 5, 6, \dots$, the following inequalities hold*:

$$g_j \ge \frac{1}{j} \sum_{k=1}^{[(j-1)/2]} \sum_{j_1 + \dots + j_m = j-m-1} \left(\frac{a}{2}\right)^{k+1} \frac{(k+1)(k+2)}{2} \frac{g_{j_1}}{j_1 + 1} \cdots \frac{g_{j_k}}{j_k + 1}.$$

Proof. Suppose that, at time $t = 0$, a particle is at the point $x(0) = 0$ and is moving with velocity ($a > 0$ is arbitrary)

$$v(t) = \frac{1}{\sqrt{1-at}} = \sum_{j=0}^{\infty} g_j t^j$$

in the field of some external force $F(x)$, which is to be found. Then

$$x(t) = \int_0^t v(s)\,ds = \left(-\frac{2}{a}\right)\sqrt{1-at} + \frac{2}{a},$$

whence we obtain

$$1 - at = \left(1 - \frac{ax}{2}\right)^2,$$

$$F = \frac{dv}{dt} = \frac{a}{2}\frac{1}{(1-at)^{3/2}} = \frac{a}{2}\frac{1}{(1-ax/2)^3}\,,$$

$$\frac{F^{(k)}}{k!} = \left(\frac{a}{2}\right)^{k+1}\frac{3\cdot 4\cdots(k+2)}{k!} = \left(\frac{a}{2}\right)^{k+1}\frac{(k+1)(k+2)}{2}\,.$$

Just as in the proof of the recurrence equations given above (the only difference being that now $v(0) = 1$ and there are no A-terms), we obtain

$$g_j = \sum_{k=1}^{[(j-1)/2]}\sum_{p=0}^{k-1}\sum_{j_1+\dots+j_m=j-m-1}\frac{1}{j}\frac{1}{k!}F^{(k)}(x_i(0))C_k^p v^p(0)\frac{g_{j_1}}{j_1+1}\cdots\frac{g_{j_{k-p}}}{j_{k-p}+1}$$

for the g_j defined above. Here, just as above, $\sum_{j_1+\dots+j_m=j-m-1}$ means summation over all $j_1,\dots,j_{k-p}$ such that

$$j_1 + \dots + j_{k-p} = j - k - 1.$$

Taking into account the fact that all the coefficients are positive and discarding the terms with $p > 0$, for all $a > 0$ we obtain

$$\begin{aligned}\frac{1}{j}&\sum_{k=1}^{[(j-1)/2]}\sum_{j_1+\dots+j_m=j-m-1}\left(\frac{a}{2}\right)^{k+1}\frac{g_{j_1}}{j_1+1}\cdots\frac{g_{j_k}}{j_k+1}\\ &\le \frac{1}{j}\sum_{k=1}^{[(j-1)/2]}\sum_{j_1+\dots+j_m=j-m-1}\left(\frac{a}{2}\right)^{k+1}\frac{(k+1)(k+2)}{2}\frac{g_{j_1}}{j_1+1}\cdots\frac{g_{j_k}}{j_k+1} \le g_j.\end{aligned}$$

Majorization. From the recurrence formula for c_{ij}, we see that these coefficients can be expressed as

$$c_{ij} = \sum_{r=1}^{d_{ij}} b_{i,j,r}N^{I_{i,j,r}}M_{i,j,r},$$

where $b_{i,j,r}$ and d_{ij} are numbers independent of either N or F, and $M_{i,j,r} \in \mathbf{A}$. In addition, by Lemma 2, we have

$$\deg M_{i,j,r} \le j - 1.$$

We shall also need some other preliminary notions. For any polynomial

$$P = \sum b_r N^{I_r} M_r,$$

where the b_r are numbers independent of either N or F, and $M_r \in \mathbf{A}$, we set

$$\beta(P) = \sum_r |b_r| N^{I_r+Q(M_r)} C_F^{Q_0(M_r)-Q(M_r)+\deg M_r},$$

where, for any monomial M_r, the natural number $Q_0(M_r)$ is equal to the sum $\sum k$ over all of its multipliers $F_{i,k,q}$. In particular,

$$\beta(c_{ij}) = \sum_r |b_{i,j,r}| N^{I_{i,j,r}+Q(M_{i,j,r})} C_F^{Q_0(M_{i,j,r})-Q(M_{i,j,r})+\deg M_{i,j,r}}.$$

For all i, k, by definition, we have

$$\beta(F_{i,k,0}) = C_F^{k+1}, \qquad \beta(\nabla^{\pm} F_{i,k,0}) = \beta(F_{i,k,1}) = C_F^{k+2} N^{-1} = C_F N^{-1} \beta(F_{i,k}).$$

In addition, the following assertion holds: for any two polynomials P_1 and P_2, we have

$$\beta(P_1 + P_2) \le \beta(P_1) + \beta(P_2), \qquad \beta(P_1 P_2) \le \beta(P_1)\beta(P_2) \tag{24}$$

and, for any monomial M, we have

$$\beta(\nabla^{\pm} M) \le (\deg M) N^{Q(M)-1} C_F^{Q_0(M)-Q(M)+\deg M+1} = (\deg M) C_F N^{-1} \beta(M), \tag{25}$$

and hence also for any polynomial P,

$$\beta(\nabla^{\pm} P) \le (\deg P) C_F N^{-1} \beta(P). \tag{26}$$

Let us call $\beta(P)$ the *majorant* of the polynomial P, because, in view of

$$|F_{i,k,1}| = |\nabla^+ F_{i,k}| \le \int_{x_i}^{x_{i+1}} |F_{i,k+1}(x)|\, dx \le C_F^{k+2} N^{-1} = \beta(F_{i,k,1}),$$

the following property holds:

$$|P| \le \beta(P).$$

It follows from (24) and (17) that

$$\beta(c_{ij}) \le \sum_{m=1}^{[(j-1)/2]} \sum_{j_1+\dots+j_m=j-m-1} \beta(A_{ij}(m; j_1, \dots, j_m)) + \sum_{k=1}^{[(j-1)/2]} \sum_{j_1+\dots+j_k=j-k-1} \beta(B_{ij}(k; j_1, \dots, j_k)).$$

Our inductive assumption (with $g_j = g_j(1)$) is as follows:

$$\beta(c_{ij}) \le \chi^j N^{5j/6-3/2} g_j, \qquad j = 1, 2, \dots, J-2. \tag{27}$$

Initial data. We can choose $\chi_0 > 0$ so that, for $j = 1, 2, 3, 4$,

$$\chi_0^j N^{5j/6-3/2} g_j \ge \beta(c_{ij}).$$

Indeed, only for $j = 3$, there is a dependence on N, but $(5/6)3 - 3/2$ is precisely 1.

Induction step for A-terms with $m > 1$. To estimate A-terms, we shall distinguish two cases: $m = 1$ and $m > 1$. For $m > 1$, we use the obvious estimates

$$\beta(\nabla^{\pm} c_{ij}) \le 2\beta(c_{ij}), \qquad \beta(\nabla^-(\nabla^+ c_{i,j_1} \cdots \nabla^+ c_{i,j_m})) \le 2^{m+1} \prod_p \beta(c_{i,j_p}).$$

Then, using (24) and (20), we obtain

$$\begin{aligned} \beta(A_{iJ}(m; j_1, \dots, j_m)) &\le \frac{m+1}{J} N^{2+m} 2^{m+1} \frac{\beta(c_{i,j_1})}{j_1+1} \cdots \frac{\beta(c_{i,j_m})}{j_m+1} \\ &\le \frac{m+1}{J} N^{2+m} 2^{m+1} \chi^{J-m-1} N^{(5/6)(J-m-1)-m(3/2)} \prod_{p=1}^{m} \frac{g_{j_p}}{j_p+1} \end{aligned}$$

$$\le 2^{m+1}\chi^{J-m-1}N^{(5/6)J-3/2}\frac{m+1}{J}\biggl(\prod_{p=1}^{m}\frac{g_{j_p}}{j_p+1}\biggr),$$

because the exponent of N admits the following estimate for $m\ge 2$:

$$2+m+\frac{5}{6}(J-m-1)-m\frac{3}{2}=\frac{5}{6}J-m\frac{8}{6}+\frac{7}{6}\le\frac{5}{6}J-\frac{3}{2}.$$

Further, by Lemma 4 with $a=2$ (if $\chi\ge 2$), we have

$$\begin{aligned}&\sum_{m=2}^{[(j-1)/2]}\sum_{j_1+\dots+j_m=j-m-1}\beta(A_{iJ}(m;j_1,\dots,j_m))\\&\quad\le N^{(5/6)J-3/2}\chi^J\sum_{m=2}^{(j-1)/2}2^{m+1}\chi^{-m-1}\sum_{j_1+\dots+j_m=j-m-1}\frac{m+1}{J}\biggl(\prod_{p=1}^{m}\frac{g_{j_p}}{j_p+1}\biggr)\\&\quad\le\biggl(\frac{2}{\chi}\biggr)^3N^{(5/6)J-3/2}\chi^Jg_j.\end{aligned}\tag{28}$$

Induction step for A-terms with $m=1$. In the case $m=1$, we shall use the following estimates for $j=J-2\ge 3$. In view of (26), (27) and Lemma 2, we have

$$\beta(\nabla^+c_{ij})\le(j-1))N^{-1}C_F\beta(c_{ij})\le(j-1))N^{-1}C_F\chi^jN^{(5/6)j-3/2}g_j$$

and, similarly,

$$\beta(\nabla^-\nabla^+c_{i,j})|\le((j-1)C_F)^2N^{-2}\chi^jN^{(5/6)j-3/2}g_j.$$

This yields the additional summand (-2) in the exponent of N, which now becomes

$$3-2+\frac{5}{6}(J-2)-\frac{3}{2}\le\frac{5}{6}J-\frac{3}{2},$$

i.e.,

$$\begin{aligned}A_{ij}(1;j_1)=A_{ij}(1;J-2)&=\frac{1}{j}|d_1|N^3\frac{\nabla^-\nabla^+c_{i,J-2}}{J-1}\\&\le 2C_F^2\chi^{j-2}N^{(5/6)J-3/2}g_{j-2}\le\frac{2C_F^2g_{j-2}}{\chi^2g_j}\chi^jN^{(5/6)J-3/2}g_j.\end{aligned}\tag{29}$$

Induction step for B-terms. For B-terms, the induction estimate is simpler, but here the degree of the monomials increases:

$$\begin{aligned}\beta(B_{ij}(k;j_1,\dots,j_k))&=\frac{1}{j}\frac{1}{k!}\beta\biggl(F^{(k)}(x_i(0))\frac{c_{i,j_1}}{j_1+1}\cdots\frac{c_{i,j_k}}{j_k+1}\biggr)\\&\le\frac{1}{j}\frac{C_F^{k+1}}{k!}\frac{1}{j_1+1}\cdots\frac{1}{j_k+1}\beta(c_{i,j_1})\dots\beta(c_{i,j_k})\\&\le\frac{1}{j}\frac{C_F^{k+1}}{k!}\frac{g_{j_1}}{j_1+1}\cdots\frac{g_{j_k}}{j_k+1}\chi^{j_1+\dots+j_k}N^{j_1+\dots+j_k};\end{aligned}$$

hence, by Lemma 4,

$$\begin{aligned}&\sum_{k=1}^{[(j-1)/2]}\sum_{j_1+\dots+j_k=j-k-1}\beta(B_{ij}(k;j_1,\dots,j_k))\\&\quad\le\frac{1}{j}\sum_{k=1}^{[(j-1)/2]}\frac{C_F^{k+1}}{k!}\sum_{j_1+\dots+j_k=j-k-1}\frac{g_{j_1}}{j_1+1}\cdots\frac{g_{j_k}}{j_k+1}\chi^{J-k-1}N^{(5/7)j}\end{aligned}$$

$$\leq C_F e^{C_F} \chi^{J-2} N^{J/2} \frac{1}{j} \sum_{k=1}^{[(j-1)/2]} \sum_{j_1+\dots+j_k=j-k-1} \frac{g_{j_1}}{j_1+1} \cdots \frac{g_{j_k}}{j_k+1}$$

$$\leq C_F e^{C_F} \chi^{J-2} N^{J/2} g_j. \tag{30}$$

Let us sum the three summands (28), (29), and (30) and choose $\chi = \chi_1 > 0$ so that

$$\left(8\chi^{-1} + \frac{2C_F^2 g_{j-2}}{g_j} + C_F e^{C_F}\right)\chi^{-2} \leq 1.$$

Then, for any $\chi \geq \max(\chi_0, \chi_1)$, we have

$$|c_{ij}| \leq \chi^J N^{J/2} g_j \leq \chi^J N^{J/2}.$$

The theorem is proved. □

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