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STABILIZATION LAWS FOR PROCESSES WITH A LOCALISED INTERACTION

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Lois de stabilisation pour processus à interactions localisées.

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Résumé.

On introduit une nouvelle classe de processus aléatoires qui sont intermédiaires entre les marches aléatoires dans $\mathbb{Z}_+^N$ et les processus à interactions locales. Cette definition a été inspirée par les réseaux de files d'attente avec plusieurs types de clients. Des nouvelles lois de stabilisation sont introduites et prouvées. Pour des marches aléatoires dans $\mathbb{Z}_+^N$, ces lois se réduisent en fait à une seule : celle obtenue comme limite d'Euler, dont l'existence est montrée pour N = 2.

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Stabilization laws for processes with a localised interaction.

Vadim Malyshev

Abstract

A class of random processes intermediate between deflected random walks and processes with a local interaction is introduced. This definition was inspired by several customer type qeueing systems. New stabilisation laws are discussed and proved. For deflected random walks in $\mathbb{Z}_+^N$ these stabilisation laws reduce to the deterministic (or Euler) limit which is proven to exist for $\mathbb{Z}_+^2$.

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1. *Introduction.*

Many examples of ergodicity conditions for one customer type queueing systems and a general viewpoint on this problem were obtained recently (see review [6]). Several customer type queueing systems seem to have completely new features which have nothing to do with one customer type queueing systems. Their discussion now is at a starting point, and the main goal of this paper is to present the new phenomena in this domain (see also [13], [14]).

Hydrodynamic limits are well understood now for many stochastic particle system models (see [1]). The main hydrodynamic space-time scalings are $\frac{x}{\varepsilon}, \frac{t}{\varepsilon}$ (leading to Euler type equations when $\varepsilon \to \infty$) and $\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}$ (leading to diffusion type equations). It is interesting enough that similar scaling limits exist for queueing systems, even very simple ones, where the space variable x is the vector of queue lengths. So one can speak about queue space-time scaling limits.

The second scaling $(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$ is well known in queueing theory, see [2, 3, 4]. It is usually called the diffusion approximation. The existence of this limit has been proven for some important queueing systems without interaction between different queues. The main trick is the construction of an appropriate martingale. Absence of interaction between queues being translated to the random walk language means that reflections from the boundaries are inherited and defined completely by the large queue behaviour.

The first scaling $(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ is less known. As it was first shown in [5] the *induced vector field* constructed formally via this limitig procedure is indispensable for obtaining necessary and sufficient ergodicity conditions. Although this scaling was implicitely used in [5] and further discussed in [6, 7] the mere existence of this scaling limit was never explicitely proved. In section 2 we fill up this gap for the two-dimensional case. Note that the semimartingale behind this proof is easily guessed. But we prefer to stress more explicitely the main steps of the proof.

There is a one-to-one correspondence between random walks in $\mathbb{Z}_+^N$ and N-queue systems with one customer type. Problems with scaling limits become much more involved for several customer types. It appears very natural to introduce a class of random processes which occupy an intermediate position between deflected random walks and processes with a local interaction, and moreover they are in the same relationship to queueing systems with several customer types as the random walks to queueing systems with one customer type. Roughly speaking, a state of the process with a local interaction (see [10]) is a string (finite or infinite) of r symbols and arbitrary local part of this string can be modified at any moment. For processes with a localised interaction introduced in section 3, only one place(or fixed subset) of a string can be modified at a time.

For the latter processes, besides these two scaling limit leading to a deterministic or diffusion motion, there exist much more delicate phenomena which we discuss here.

We consider mainly one-dimensional case (transient case in section 3 and ergodic case in section 4), but formulate also some conjectures for 2-dimensional case in section 5.

2. *Deterministic motion for deflected two-dimensional random walks.*

One-dimensional deflected random walks.

Begin with a one-dimensional example. Consider the discrete time chain $\xi_j(t)$ on Z_+ starting from j and having jump probabilities $p_{i,i-1} = p$ and $p_{i,i+1} = q = 1 - p$, $p > q$ (from 0 it jumps to 1 with probability 1). Take the starting point at $j = [nx]$ sufficiently large, then the law of large numbers (LLN) and some simple arguments show that $\frac{\xi_{[nx]}([\tau n])}{n}$, $n \to \infty$, converges in probability for any fixed positive real τ and x to a constant

$$\max \{0, x + \tau(q-p)\},$$

and thus describes the deterministic motion with the constant velocity $q - p$. If it reaches the origin it stays there forever.

Equivalently one can take ε-lattice $Z_{+,\varepsilon} = \{0, \varepsilon, 2\varepsilon,..\}$ with the starting point at x, thus providing the space-scale ε^{-1}, the random walk jumps on $-\varepsilon$ with probability p and on ε with probability q. Then during the time $\frac{\tau}{\varepsilon}$ (thus scaling the time also like ε^{-1}) we shall be approximately at the point $x + \tau(q - p)$ (this becomes exact in the $\varepsilon \to 0$ limit).

Two-dimensional deflected random walks.

Recall some definitions related to deflected random walks in $\mathbb{Z}^2_+ = \{(z_1, z_2) : z_i \geq 0$ are integers$\}$. Consider a discrete time homogeneous Markov chain $\mathcal{L}$, which is assumed to be irreducible and aperiodic unless otherwise stated,with the set of states $\mathbb{Z}^2_+$.

For any $\Lambda \subset \{1,2\}$ we define the face B^Λ of $\mathbb{R}^2_+ = \{r = (r_1, r_2) : r_i \geq 0$ are real$\}$ by

$$B^\Lambda = \{r = (r_1, r_2) : r_i > 0,\ i \in \Lambda;\ r_i = 0,\ i \notin \Lambda\}.$$

Sometimes we shall write Λ instead of B^{Λ} whenever confusion cannot occur.
Transition probabilities $p_{\alpha\beta}$ of $\mathfrak{L}$ satisfy the following conditions :
Condition B (boundedness of jumps):

$$p_{\alpha\beta} = 0 \text{ for } \| \alpha - \beta \| > d,$$

where $d > 0$ and $\| \alpha \| = \max_i | \alpha_i |$, $\alpha = (\alpha_1, \alpha_2)$.
Condition H (space homogeneity):
for any Λ and for any $a \in B^{\Lambda} \cap \mathbb{Z}_+^2$,

$$p_{\alpha\beta} = p_{\alpha+a,\beta+a}$$

for all $\alpha \in B^{\Lambda} \cap \mathbb{Z}_+^2$, $\beta \in \mathbb{Z}_+^2$. So one can write $p_{\alpha\beta} = p(\Lambda; \beta - \alpha)$.

Note that it follows from Condition H that $p_{\alpha\beta} = 0$ if $\beta_i - \alpha_i < -1$ for some i.
Define now the induced chain $\mathfrak{L}^{\{1\}}$, corresponding to the face $\{1\}$. Choose an arbitrary point $a \in B^{\{1\}} \cap \mathbb{Z}_+^2$ and draw a line perpendicular to $B^{\{1\}}$ and containing the point a. The intersection of this line with $\mathbb{Z}_+^2$ is denoted by $C^{\{1\}}$.

Transition probabilities ${}_1p_{\alpha\beta}$ for the induced chain $\mathfrak{L}^{\{1\}}$ are equal to

$${}_1p_{\alpha\beta} = p_{\alpha\beta} + \sum_{\gamma\neq\beta} p_{\alpha\gamma}$$

where summation is performed over all $\gamma \in \mathbb{Z}_+^2$ such that the straight line connecting γ and β is perpendicular to $C^{\{1\}}$. It is important to note that this construction does not depend on a.
The face $\{1\}$ is called ergodic if $\mathfrak{L}^{\{1\}}$ is ergodic. It is equivalent to the fact that the vector (1) defined below has its second coordinate negative.
The induced chain $\mathfrak{L}^{\{2\}}$, corresponding to the face $\{2\}$, is defined in a similar way.
Define now the vector field $v(x)$ in $\mathbb{R}_+^2 - \{0\}$:
(i) $v(x)$ is constant inside $B^{\{1,2\}}$ and equal to

$$M(\alpha) = \sum_{\beta} (\beta - \alpha) p_{\alpha\beta} \tag{1}$$

for any point $\alpha \in B^{\{1,2\}} \cap \mathbb{Z}_+^2$;

(ii) $v(x)$ is constant inside $B^{\{1\}}$ and equal to (1) if $\{1\}$ is not ergodic; if $\{1\}$ is ergodic then $v(x)$ is also constant inside $B^{\{1\}}$ but its coordinates are equal to

$$v_2(x) = 0, \tag{2}$$

$$v_1(x) = \sum_{\beta \in C^{\{1\}}} (\beta - \alpha)_1 \pi_\alpha , \tag{3}$$

where π_α are the stationary probabilities of $\mathfrak{L}^{\{1\}}$.

(iii) on $B^{\{2\}}$ $v(x)$ is defined similarly.

Assumption O_1. We always assume that both coordinates of (1) and also (3) are nonzero, similarly for (iii).

All possible cases are given on Fig. 1. The following theorem was proved in [5].

Theorem 1. Under the assumption O_1 the random walk is ergodic iff the dynamical system defined by the second vector field reaches zero from every point. In other words only cases 7, 2 and symmetric to 2 are ergodic.

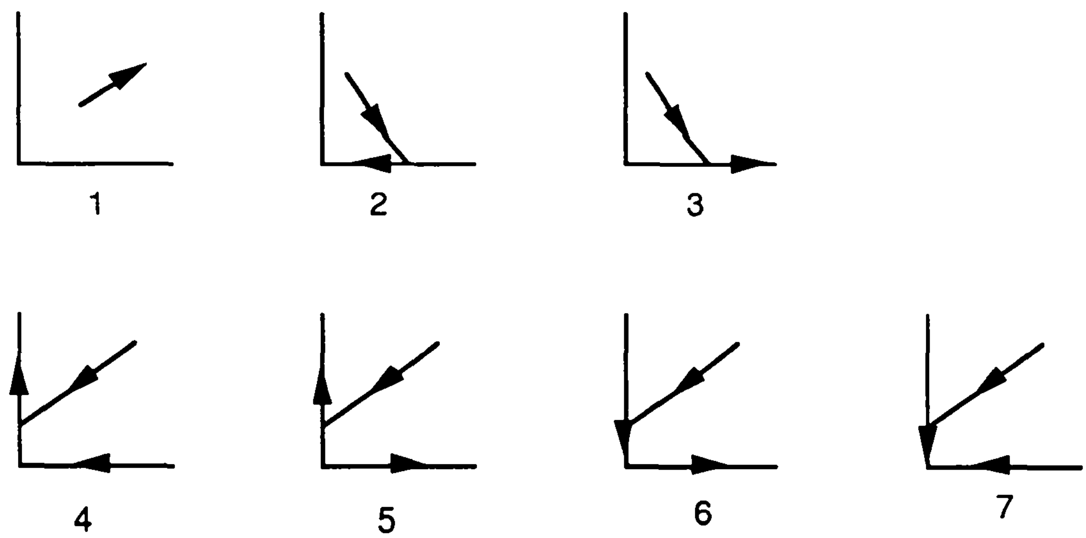

Fig. 1

Let us consider Fig. 2 (or Fig. 1.2) in more detail, where the fat lines show the second vector field, ordinary lines show vectors $M(\alpha)$.

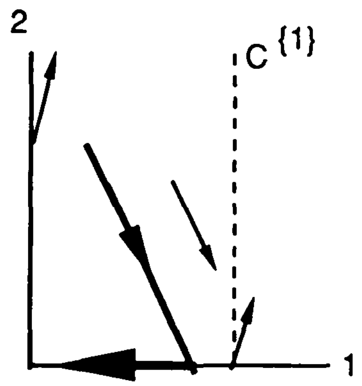

Fig. 2

The induced chain $\mathfrak{L}^{\{1\}}$ is ergodic as it is a random walk on $\mathbb{Z}_+$ with positive drift. Its stationary probabilities are easily calculated and so also $v^{\{1\}}$. By similar arguments $\mathfrak{L}^{\{2\}}$ is transient.

Deterministic limit.

For the case of Fig.2 consider the following dynamical system $T_t : \mathbb{R}^2_+ \to \mathbb{R}^2_+$.

Starting from a point $x = (x_1, x_2)$, $x_1, x_2 > 0$, we move with the speed $v(x)$ until we reach axis 1 and then we move with the speed v_1 along this axis; if $v_1 < 0$ then after reaching 0 we stay in it forever. We formulate and prove the theorem for the case of Fig. 2. The other cases of Fig.1 can be reformulated and proved in a similar way.

Theorem 2. The random walk $\xi_{[nx]}([nt])$ starting from $[nx]$ after time $[tn]$ has the following Euler limit : for all $x \in \mathbb{R}^2_+$ and $t \in \mathbb{R}_+$ in probability

$$\lim_{n\to\infty} \frac{\xi_{[nx]}([nt])}{n} = T_t x \tag{4}$$

Proof. Consider first the random walk $\eta(s)$ in $\mathbb{Z}^2$ starting from $[nx]$ with the transition probabilities equal to those of $\xi(s)$ inside $\mathbb{Z}^2_+$. Let $M = (M_1, M_2)$ be the mean jump vector for $\eta(s)$ (equal to that of $\xi(s)$ inside $\mathbb{Z}^2_+$). Then the LLN gives

$$\frac{\eta([tn])}{n} \to x + tM \in \mathbb{R}^2_+ \tag{5}$$

in probability. We need two more inequalities for $\eta(s)$ considered as the sum of i.i.d. random vectors :

(i) (Kolmogorov's exponential bounds, see [8], p. 254) :

let X_n be i.i.d. random variables with mean 0 and variance 1, put

$$c = \max_{1 \le k \le n} \left| \frac{X_k}{\sqrt{n}} \right|, \quad S_n = X_1 + \ldots + X_n ,$$

then

$$P(S_n > \sqrt{n}\, r) < \exp\left(-\frac{1}{4} r^2\right), \quad \text{if } \; r c \le 1, \qquad (6)$$

$$P(S_n > \sqrt{n}\, r) < \exp\left(-\frac{1}{4}\frac{r}{c}\right), \quad \text{if } \; r c \ge 1. \qquad (7)$$

So in particular, taking $r = n^{\frac{\varepsilon}{2}}$ and using (6) componentwise, together with $|X_k| \le$ const, we get

$$P\left(\left| \eta([\, tn + un^{\frac{1}{2}+\varepsilon}]) - (tn + un^{\frac{1}{2}+\varepsilon}) M \right| > n^{\frac{1}{2}+\frac{\varepsilon}{2}} \right) < \exp(- C n^{\varepsilon}) \qquad (8)$$

for some C depending on u and on the random walk in question ;

(ii) (Kolmogorov's inequality) : if $t < \frac{x_2}{M_2}$, then

$$P (\eta(s) \text{ intersects the axis 1 for some } s \in [0, tn]) < \text{const } n^{-1} ; \qquad (9)$$

if $t = \frac{x_2}{M_2}$, $u < 0$, then

$$P (\eta(s) \text{ intersects the axis 1 for some } s \in [0, tn + un^{\frac{1}{2}+\varepsilon}]) < \text{const } n^{-2\varepsilon} ; \qquad (10)$$

Now to prove (4) for $t < \frac{x_2}{M_2}$, we use (8) outside of the event (9) ; for $t = \frac{x_2}{M_2}$ we take $u = -1$ in (10) and so with the probability close to 1 we can consider $\eta(s)$ instead of $\xi(s)$ and (4) follows together with the inequality

$$P\left(\left| \xi_{[nx]}([nt]) - \left(x + \frac{x_2}{M_2} M\right) \right| > c\, n^{\frac{1}{2}+\varepsilon} \right) < \text{const } n^{-2\varepsilon} .$$

Let us take now $t > \frac{x_2}{M_2}$, put $\tau = s - [n\frac{x_2}{M_2}]$.

It is convenient for the proof to consider the following general scheme. Let ζ_τ be an ergodic Markov chain (in our case it is the induced chain for the axis 1) with the state space $\mathfrak{A}$ (in our case it is $\mathbb{Z}_+$), defined on a probability space (Ω, Σ, μ). Also let us consider on some probability space $(\Omega_1, \Sigma_1, \mu_1)$ mutually independent random variables

$$g_\tau(\alpha_1, \alpha_2) = g_{\tau, \alpha_1, \alpha_2}(\omega_1), \alpha_i \in \mathfrak{A}, \;\; \tau = 0, 1, \ldots ; \omega_1 \in \Omega_1, \; |g_\tau(\alpha_1, \alpha_2)| \le d,$$

indexed by τ and pairs (α_1, α_2). Distribution of $g_\tau(\alpha_1, \alpha_2)$ does not depend on τ and for our case $g_\tau(\zeta_\tau, \zeta_{\tau+1})$ is the 1- component of the jump of the random walk $\xi(s)$ from the point $(n_1, n_2) \in \mathbb{Z}_+^2$ where $n_1 > 0$ is arbitrary and $n_2 \in \mathcal{Q} = \mathbb{Z}_+$ is equal to ζ_τ, conditioned on the jump $\zeta_\tau \to \zeta_{\tau+1}$ (perpendicular to the axis 1) of the induced chain. Let us put on $\Omega \times \Omega_1$

$$S_n = B + \sum_{\tau=0}^{n-1} g_\tau (\zeta_\tau, \zeta_{\tau+1}),$$

where $| B - n (x_1 + \frac{x_2}{M_2} M_1) | < c n^{\frac{1}{2}+\varepsilon}$, $|\zeta_0| < c n^{\frac{1}{2}+\varepsilon}$.

We should now wait for not more than $n^{1-\delta}$ units of time until ζ_τ reaches zero. Then we could use Kolmogorov's inequalities in a similar fashion for this functional on the Markov process.

However we shall show now that this study can be reduced to i.i.d. random variables with exponential tales.We can assume that $\zeta_0 = 0$, consider all random times $0 = \tau_0, \tau_1, \tau_2, \ldots$, when $\zeta_{\tau_i} = 0$. Let us put

$$y_i = S_{\tau_i} - S_{\tau_{i-1}}$$

It is easy to prove that :

(i) $E\, y_i = (\pi_0)^{-1} v_1$, where π_0 is the stationary probability of 0 for the induced Markov chain ;

We shall use notations from [9], p.52, where ${}_ip_{ij}^{(k)}$ is the probability, starting from i, to reach j after k steps without hitting i any more during these k steps. As ${}_ip_{ij}^{*} = \frac{\pi_j}{\pi_i}$ we have

$$E\, y_i = \sum_{k,j} Eg(j)\, {}_ip_{ij}^{(k)} = \sum_j Eg(j)\, {}_ip_{ij}^{*} = \sum_j Eg(j) \frac{\pi_j}{\pi_i} = (\pi_i)^{-1} v_1$$

(ii) there exist constants a, b > 0 such that

$$P(| y_i | > k) < a \exp(- b k).$$

Now we shall prove that

$$P(\frac{| S_n |}{\sqrt{n}} > n^{\frac{1}{2}-\varepsilon}) < \exp (- \frac{1}{4} n^{\frac{1}{2}-\varepsilon-\delta}) \qquad (11)$$

Choose a suitable truncation of the random variable X_k

$$\hat{X}_k = X_k \mathbf{1}(| X_k | \leq n^{\frac{1}{2}+\delta})$$

Then

$$P\,(\text{ at least one } \mid X_k \mid \; > n^{\frac{1}{2}+\delta}) \;\le\; an \exp(-\,b\, n^{\frac{1}{2}+\delta}).$$

We have

$$c = \max_{1\le k\le n} \left| \frac{\hat{X}_k}{\sqrt{n}} \right| \;\le\; n^{\delta}\,.$$

So (11) follows from (7). ■

3. Processes with a localised interaction.

Simplest processes (generalised random walks in $\mathbb{Z}_+$).

Consider a discrete time homogeneous countable Markov chain $\xi(t)$ with the set of states

$$\mathbb{W}(1; r) = \bigcup_{n=0}^{\infty} \{1,\dots,r\}^n.$$

In other words, states are strings $\alpha = (x_n,\dots, x_1)$ with r symbols of arbitrary length $n = n(\alpha)$. We denote by $\emptyset$ an empty string (of length 0). If we have two strings $\alpha = (x_n,\dots, x_1)$ and $\beta = (y_m,\dots, y_1)$ we define their composition (of length m + n) by

$$\alpha\beta = (x_n,\dots, x_1, y_m,\dots, y_1).$$

For the transition probabilities $p_{\alpha\beta}$, $\alpha \to \beta$, we assume the following conditions.

Condition B (boundedness of jumps): for some $d < \infty$

(i) if $n(\alpha) < d$ then $p_{\alpha\beta} \neq 0$ only if $n(\beta) \leq d + n(\alpha)$;

(ii) if $n(\alpha) \geq d$ then $p_{\alpha\beta} \neq 0$ only if $\alpha = \gamma\theta$, $\beta = \delta\theta$ for some γ, δ, θ with $n(\gamma) = d$, $n(\delta) \leq d + n(\gamma)$.

Condition H (space homogeneity): if $n(\gamma) = d$ then $P(\xi(t+1) = \gamma\theta \mid \xi(t) = \delta\theta)$ does not depend on θ but only on γ and δ. So we can denote this conditional probability by $q(\gamma,\delta)$: we delete γ from the left and append δ instead.

Together with the just defined countable Markov chain $\xi(t)$ whose states are finite strings we consider two other random processes $\xi^1(t)$ and $\xi^2(t)$ on semiinfinite strings. More exactly their sets of states are

$$\mathcal{A}_1 = \{(k; x_k, x_{k+1},\dots), k \in \mathbb{Z}, x_i \in \{1,\dots,r\}\},$$

$$\mathcal{A}_2 = \{(x_0, x_1,\dots), x_i \in \{1,\dots,r\}\} = \{1,\dots,r\}^{\mathbb{Z}_+},$$

correspondingly. Define the map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ by $\phi(k; x_k, x_{k+1},\dots) = (y_0, y_1, \dots)$ with $y_0 = x_k$, $y_1 = x_{k+1}, \dots$. Transitions for $\xi^1(t)$ are the same as for $\xi(t)$, i.e. $\gamma\delta \to \delta\delta$ (for semiinfinite δ). We define $\xi^2(t)$ on the same probability space as $\xi^1(t)$ just by renumerating: $\xi^2(t) = \phi(\xi^1(t))$.

For a probability measure μ on $\mathcal{A}_2$ we define the correlation functions

$$p(i; \alpha) = p_\mu(i; \alpha) = \mu(\{x_i = \alpha_0,\dots, x_{i+n-1} = \alpha_{n-1}\})$$

for $\alpha = (\alpha_0,\dots, \alpha_{n-1})$, $i \in \mathbb{Z}_+$. Put $p(\alpha) = p(0; \alpha)$. A probability measure μ is called translation invariant if $p(i; \alpha) = p(\alpha)$ for all α, i.

Note that by compactness there exists at least one stationary measure for the process $\xi^2(t)$.

Condition ND (nondegeneracy) : for any string γ, $n(\gamma) = 1$, there exisis k such that for all α

$$p^{(k)}_{\gamma\alpha,\alpha},\ p^{(k)}_{\alpha,\gamma\alpha} > 0$$

Now we give the central definition.

Stabilization laws (transient case). Let $\xi(t)$ be transient. We say that the stabilisation laws T1, T2, T3 hold if correspondingly :

(T1) for any real function f on $\{1,...,r\}$ the following limit exists in probability

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} f(\xi_{n(t)}(t)) = v_f$$

where v_f is a constant depending on f, where $\xi_{n(t)}(t)$ is the leftmost symbol of the string $\xi(t)$.

(T2) there exists the unique stationary measure μ of the process $\xi^2(t)$ and for any k, $\alpha_0,...,\alpha_{k-1}$,

$$P((\xi_{n(t)}(t) = \alpha_0,..., \xi_{n(t)-k+1}(t) = \alpha_{k-1}) \to p_\mu(\alpha_0...\alpha_{k-1}) \quad \text{if} \quad t \to \infty.$$

(T3) for any $m(t) \to \infty$, $m(t) \le n(t)$,

$$P((\xi_{m(t)}(t) = \alpha_0,..., \xi_{m(t)-k+1}(t) = \alpha_{k-1}) \to p_\mu(\alpha_0...\alpha_{k-1}) \quad \text{if} \quad t \to \infty.$$

We prove here T2 and show that T1 is its corollary.

Condition $L_\pm$ (Lyapounov function) n(.) is a Lyapounov function for the Markov chain $\xi(t)$. In other words, there exist $k > 0$, $\varepsilon > 0$ such that for all α except a finite number

$$\sum_\beta p^{(k)}_{\alpha\beta} n(\beta) - n(\alpha) > \varepsilon \quad (L_+ \text{ condition})$$

$$\sum_\beta p^{(k)}_{\alpha\beta} n(\beta) - n(\alpha) < -\varepsilon \quad (L_- \text{ condition})$$

where $p^{(k)}_{\alpha\beta}$ are k-step transition probabilities.

Remark 1. L_- implies ergodicity, L_+ implies transience by well-known criteria (see [5]). Let for $n(\gamma) = d$

$$q_-(\gamma) = \sum_{\delta : n(\gamma) > n(\delta)} q(\gamma,\delta), \qquad q_+(\gamma) = \sum_{\delta : n(\gamma) < n(\delta)} q(\gamma,\delta).$$

L_+ condition holds for example, if for any γ

$$q_-(\gamma) < q_+(\gamma) \tag{1}$$

In this case one can take $k = 1$ in the condition $L_\pm$.

Theorem 3. Let the conditions **B, H, ND, L$_+$** hold. Then $\xi(t)$ is transient and the stabilisation principle T2 holds.

Proof. Let $\mathbb{P} = (p_{\alpha\beta})$, $p_{\alpha\beta} = P(\xi(t+1) = \beta \mid \xi(t) = \alpha)$, be the stochastic operator for the Markov chain $\xi(t)$. So if we introduce the row vector $p_t = (P(\xi(t) = \alpha), \alpha \in \mathcal{Q})$ then

$$p_{t+1} = p_t \mathbb{P} \tag{2}$$

Introduce the row vector of correlation functions $p = (p(\alpha), \alpha \in \mathcal{Q})$. Then a stationary measure μ of the process $\xi^2(t)$ has to satisfy the following equation

$$p = p\,\mathbb{P}_1 + c \tag{3}$$

where $\mathbb{P}_1$ is called the principal part of $\mathbb{P}$ and $c = (c(\alpha), \alpha \in \mathcal{Q})$ is the vector. We define them by writing down explicitely these equations (for simplicity we write them for $d = 1$). For any symbols $\alpha_0, \alpha_1, \beta_0 \in \{1,\dots,r\}$

$$p(\alpha_0) = \sum_{\alpha_1} p(\alpha_1)[\, q(\alpha_1, \alpha_0) + \sum_{\beta_0} q(\alpha_1, \alpha_0\beta_0)] + \sum_{\alpha_1} p(\alpha_1\alpha_0)\, q(\alpha_1; \emptyset) \tag{4}$$

If $n(\gamma) \geq 1$, $\gamma = (\gamma_0, \dots, \gamma_{n-1})$, then

$$p(\alpha_0\gamma) = \sum_{\alpha_1} p(\alpha_1\gamma)\, q(\alpha_1, \alpha_0) + \sum_{\alpha_1} p(\alpha_1\gamma_1\dots\gamma_{n-1})\, q(\alpha_1, \alpha_0\gamma_0) + \sum_{\alpha_1} p(\alpha_1\alpha_0\gamma) q(\alpha_1; \emptyset) \tag{5}$$

Equations (4)-(5) form a coupled infinite system of equations. We can rewrite (5) as

$$p(\beta) = \sum_{\alpha} p(\alpha)\, p_{\alpha\beta}, \quad n(\beta) \geq 2,$$

i.e. it coincides with (2). But (4) has the form, $n(\beta) = 1$,

$$p(\beta) = \sum_{\alpha} p(\alpha)\, p_{\alpha\beta} + \sum_{\alpha:n(\alpha)=1} p(\alpha)\, \overset{+}{p}_{\alpha\beta} + c(\beta)$$

where $c(\beta) = 0$ for $n(\beta) \geq 2$ and for $n(\beta) = 1$

$$c(\beta) = \min_{\alpha} \sum_{\beta_0} q(\alpha, \beta\beta_0), \quad n(\alpha) = 1, \tag{6}$$

$$\overset{+}{p}_{\alpha\beta} = \sum_{\beta_0} q(\alpha;\beta\beta_0) - c(\beta)$$

is the probability that the string α of length $n = n(\alpha)$ increases its length so that the symbol β appears on the left side.

Iterating (3) we get

$$p = c\,(1 + \mathbb{P}_1 + \dots + \mathbb{P}_1^{n-1}) + p\,\mathbb{P}_1^n \tag{7}$$

Let us denote matrix elements of $\mathbb{P}_1^n$ by $p_1^{(n)}(\alpha, \beta)$. For given β, let for example $n(\beta) = 1$,

$$\min_{\alpha:n(\alpha)=k} p_1^{(n)}(\alpha, \beta) = \phi_{k,n}(\beta) \to 0$$

if $\max(k, n) \to \infty$, and moreover

$$\phi_{k,n}(\beta) \leq C(\beta)\, \chi^{\max(k, n)} \tag{8}$$

where $0 < \chi < 1$ does not depend on β. To prove (8) note that the bound $C(\beta)\,\chi^k$ follows from the existence of the linear Lyapounov function by the exponential bound of Lemma 1.1, [5].

To prove the bound $C(\beta)\chi^n$ we define the process $\xi^3(t)$ by modifying the process $\xi(t)$ in the following way :we append an absorbing state 0 to which we can enter only from β with $n(\beta) = 1$, with the probability $p_{\beta 0} = q(\beta;\emptyset) + c(\beta)$. Then this bound follows from the obvious inequality $p_1^{(n)}(\alpha, \beta) \leq \Pr(\xi_3(t)$ starting from α is not killed).

From (8) it follows that β-component of the vector $p\,\mathbb{P}_1^n$ has the bound

$$|(p\,\mathbb{P}_1^n)(\beta)| \leq \sum_{\alpha} p(\alpha)p_1^{(n)}(\alpha,\beta) \leq \sum_{k} \sum_{\alpha:n(\alpha)=k} p(\alpha)p_1^{(n)}(\alpha,\beta) \leq$$

$$\leq \sum_{k} C(\beta)\,\chi^{\max(k,n)} \leq C'(\beta)\,\chi^n.$$

So the stationary measure of $\xi^2(t)$ is unique and is given by

$$p = c\sum_{n=0}^{\infty} \mathbb{P}_1^n$$

Now we proceed to the convergence proof. We shall use the coupling method. We consider two independent copies of $\xi(t)$: $\xi(t)$ and $\xi'(t)$, but with different initial conditions.

Due to our conditions there exist $\alpha_0, \ldots, \alpha_k$ such that for the process $\xi(t)$ there exists a sequence $\tau_1, \tau_2, \ldots$ of random times which is infinite a.s. and has the following properties :

1. $\xi_{n(\tau_j)}(\tau_j) = \alpha_0, \ldots, \xi_{n(\tau_j+k-1)}(\tau_j+k-1) = \alpha_k$;
2. $n(\tau_j + i) = n(\tau_j + i - 1) + 1$, $i = 1, \ldots, k-1$;
3. $n(t) \geq n(\tau_j)$ for all $t > \tau_j$.

Let τ_j' be the similar random times for the process $\xi'(t)$. Then with probability 1 there exist i and j such that $\tau_i = \tau_j'$. After such event the behaviour of $\xi(t)$ and $\xi'(t)$ does not depend on the past and we couple $\xi(t)$ and $\xi'(t)$ at these moments of time. By standard arguments $\xi(t)$ and $\xi'(t)$ become identical with probability 1 and moreover exponentially fast. We can use the same method to couple $\xi(t)$ and the stationary process $\xi^2(t)$. This gives the desired result. ■

Corollary 1. Under conditions of theorem 3 , $\frac{n(t)}{t}$ converges as $t \to \infty$ in probability to a constant v. For example for d = 1 it is equal to

$$v = \sum_{\alpha_1} p_\mu(\alpha_1) \Big(\sum_{\alpha_0,\beta_0} q(\alpha_1,\alpha_0\beta_0) - q(\alpha_1,\emptyset) \qquad (9)$$

Proof. As $E\,n(t) \sim vt$ we must only prove that $D\,n(t) \sim Ct$ for some $C > 0$. Put

$$n(t) = \sum_{\tau=0}^{t-1} s(\tau)\,, \quad s(\tau) = n(\tau+1) - n(\tau)$$

The estimate of the covariance

$$|\langle s(\tau)\,,\ s(\tau')\rangle| \equiv |\,E\, s(\tau)\, s(\tau') - (E\, s(\tau))\ (E\, s(\tau'))\,| < C\,\lambda^{|\tau'-\tau|}$$

is a byproduct of the coupling argument above. ◆

4. Ergodic case.

The ergodic case for the processes with localised interactions is less evident because this depends strongly on the initial string (in transient cases the initial state becomes irrelevant exponentially fast). The most interesting stabilisation laws take a conditional form : for example, if *in the stationary state* the queue is large then its movement back to zero is stationary deterministic.

Stabilisation laws (ergodic case). Let $\xi(t)$ be ergodic. We say that the stabilisation laws E1, E2, E3 hold if correspondingly :

(E1) assume we start at time 0 from the stationary state with queue length $[xN]$, x and t being positive reals, then in probability

$$\frac{n([tN])}{N} \to U_t x,$$

where U_t is the deterministic motion to 0 with the velocity v (given by formula (9) of the previous section) , it stays in 0 forever when it reaches it.

(E2) let $\xi(0)$ be a random string $(\eta_N,..., \eta_1)$ of the fixed length N, its distribution being inherited $(\eta_N = \eta(N),..., \eta_1 = \eta(1))$ from some stationary ergodic process $\eta(t)$ with values in $\{1,...,r\}^{\mathbb{Z}}$. Then,

(i) as $t_1 < t_2 = \varepsilon N$ for ε sufficiently small and t_1 tends to infinity (so and N does), then finite-dimensional distributions of $\xi(t)$, $t \in [t_1, t_2]$, tend to those of a stationary process;

(ii) Let τ_n be the first reaching time of the rightmost substring $(\eta_n,..., \eta_1)$ of η, then for any function $f : \{1,...,r\} \to \mathbb{R}$,

$$\lim \frac{1}{N-n} \sum_{t=1}^{\tau_n} f(\xi(t)) \to v_f \quad \text{in probability}$$

as N and N – n tend to infinity, v_f being a constant.

(E3) for any fixed k, $\alpha_0,..., \alpha_{k-1}$, $n \leq N$, the conditional stationary distribution

$$\pi(\{\beta = (\beta_N,...,\beta_1)' : \beta_n = \alpha_0,..., \beta_{n-k+1} = \alpha_{k-1}\} \mid n(\beta) = N) \qquad (1)$$

$$= \frac{\sum_{\beta_1,\dots,\beta_{n-k},\beta_{n+1},\dots,\beta_N} \pi(\beta = (\beta_N,\dots,\beta_1) : \beta_n = \alpha_0,\dots, \beta_{n-k+1} = \alpha_{k-1})}{\sum_{\beta=(\beta_1,\dots,\beta_N)} \pi(\beta)} \rightarrow$$

$\rightarrow$ to some $p(\alpha_0,\dots, \alpha_{k-1})$,

when n tends to infinity (and so N does). These $p(\alpha_0,\dots, \alpha_{k-1})$ give rise to a stationary process on $\{1,\dots,r\}^{\mathbb{Z}_+}$.

We shall prove the stabilisation law E3 and E2(ii).

Theorem 4. If $\xi(t)$ is ergodic then the stabilisation laws E3 and E2(ii) hold.

Proof of E3. For all $n \neq 0$ the equations for the stationary probabilities have the form

$$\Pi_n = \sum_{i=-1}^{1} \Pi_{n-i} Q_i \tag{2}$$

where Π_n is the vector of stationary probabilities for strings of lengths n, n+1,..., n+d. Q_i is the matrix of $q(\gamma,\delta)$.

We shall use some ideas from mathematical statistical physics, see [11].

Paths.

Define a path Γ as a sequence of strings $\alpha^0,\dots, \alpha^M$ such that $|n_i - n_{i-1}| \le d$, where $n_i = n(\alpha^i)$, $i = 1, \dots, M$ and where α_i is obtained from α_{i-1} by deleting γ_i and appending δ_i from the left. We write $\Gamma(N)$ for a path with $n_M = N$, let $\alpha(\Gamma) = \alpha^M$ be the last string of Γ, $\alpha^0(\Gamma)$ - the first string of Γ. Otherwise speaking we could define a path as an admissible sequence of pairs (γ_i,δ_i), $i = 1, \dots, M$, so that there exists a sequence of strings $\alpha^0,\dots, \alpha^M$, where α_i is obtained from α_{i-1} by deleting γ_i and appending δ_i from the left. The contribution of Γ is defined as

$$q(\Gamma) = \prod_{i=1}^{M} p_{\alpha_{i-1},\alpha_i},$$

where $p_{\alpha_{i-1},\alpha_i} = q(\gamma_i, \delta_i)$ if $n(\alpha_{i-1}) \ge d$.

Then for $n(\alpha) > d$

$$\pi(\alpha) = \sum_{\Gamma} \pi(\alpha^0(\Gamma))q(\Gamma) \tag{3}$$

where the summation is over all Γ such that $\alpha(\Gamma) = \alpha$, $n(\alpha^0(\Gamma)) < d$, $n(\alpha^i) \ge d$, $i = 1,\dots, M-1$. This follows from (2) by iteration.

Gibbs formula.

Substituting (3) to (1) we get

$$\pi(\{\beta = (\beta_N,\dots,\beta_1) : \beta_n = \alpha_0,\dots, \beta_{n-k+1} = \alpha_{k-1}\} \mid n(\beta) = N) =$$

$$= \frac{\Sigma^{\alpha,n} q(\Gamma)}{\Sigma q(\Gamma)} \qquad (4)$$

where in the denominator the summation is over all $\Gamma = \Gamma(\beta)$ with $n(\beta) = N$, in the numerator the summation is over all $\Gamma = \Gamma(\beta)$, $\beta = (\beta_N,\dots,\beta_1)$, with $n(\beta) = N$ and $\beta_n = \alpha_0,\dots, \beta_{n-k+1} = \alpha_{k-1}$.

Resummation.

We want to make resummations in formula (3). To do this some technical definitions are necessary.

Define blocks $I_0, I_1,\dots$ of $\mathbb{Z}_+$ as the intervals $I_k = [dk, d(k+1))$. We say that a string α terminates at the block I_k ($\alpha \in I_k$) if $n(\alpha) \in I_k$. Define I_k-part $\alpha(I_k)$ of α as its leftmost part with indices in I_k. For any two $\alpha,\beta \in I_k$ let us put

$$Q(\alpha\to\beta) = \sum_{\Gamma} q(\Gamma),$$

where the sum is over all $\Gamma = (\alpha^0,\dots, \alpha^M)$ with $\alpha^0(\Gamma) = \alpha$, $\alpha(\Gamma) = \beta$, $n(\alpha^i) \geq dk$ (i.e. α^i do not belong to blocks with smaller indices) for all i.

Lemma 1. $Q(\alpha\to\beta) < c < \infty$ where (by homogeneity) c does not depend on α, β, k.

Proof. $Q(\alpha\to\beta)$ is the mean number of times of hitting β without visiting blocks $I_0,\dots,I_{k-1}$ before.It is bounded by the mean time of reaching some state in $I_0 \cup\dots\cup I_{k-1}$. ■

Transfer-matrix.

For any $\Gamma = \Gamma(\beta)$, $\beta \in I_N$, and any $j < N$ define

$$b_j = \max_i \{ i : \alpha^i \in I_j \}.$$

Then the contribution of Γ can be written as

$$q(\Gamma) = \prod_j q(\Gamma(b_j+1, b_{j+1}))p_{b_{j+1},b_{j+1}+1} \qquad (5)$$

where $\Gamma(j,k)$, $j < k$, is the path $\alpha^j\dots\alpha^k$. Note that $b_j+1 \in I_{j+1}$. Moreover, matrix elements of $Q(\alpha\to\beta) = Q$ and of p\S\DO5($b_{j+1},b_{j+1}+1$) = P depend only on $\alpha(I_k)$, $\beta(I_k)$ and $\alpha^{b_{j+1}}(I_{j+1})$, $\alpha^{b_{j+1}+1}(I_{j+2})$. So the rows and columns of Q and P are numerated by all strings of length 1,...,d and so they are defined in the finite-dimensional space $\mathfrak{B}$ (the linear space of functions on these strings, dimension D is $r + \dots + r^d$).

Lemma 2. In (4) we have

$$\sum q(\Gamma) = (\delta_N, \rho A^{N-1}) \qquad (6)$$

where $A = QP$ is a finite dimensional matrix, δ_N is the function in $\mathfrak{B}$ equal to 1 for strings of length N (mod d) and O otherwise, ρ is the vector numerated by I_1-parts γ

$$\pi_\gamma = \sum_\alpha \sum_{\beta: I_1(\beta)=\gamma} \pi(\alpha) p_{\alpha\beta}$$

Proof . ◆

Let $e_1, \dots, e_D$ be the eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_D$, $\lambda_1 > |\lambda_i|$, $i = 2,\dots,$ D, $\lambda_1 > 0$. If $\rho = \sum_j c_j e_j$. Note that e_1 is a positive eigenvector so $c_1 > 0$ by Perron-Frobenious theorem, and

$$(\delta_N, \rho A^{N-1}) \sim \lambda_1^{N-1}(\delta_N, c_1 e_1). \qquad (7)$$

Note that ergodicity is equivalent to the fact that $\lambda_1 < 1$, because

$$\pi(\beta) = \rho_{\beta_1} a_{\beta_1\beta_2} \cdots a_{\beta_{N-1}\beta_N}, \quad n(\beta) = N.$$

Similarly we can get asymptotics for the numerator in (4) (for simplicity for the case k = 1, so $\alpha = \alpha_0$). Let $n \in I_s$, δ_α be the function in $\mathfrak{B}$ equal to 1 on all γ with α on n(mod d)-place, and 0 otherwise. Then

$$\sum^{\alpha,n} q(\Gamma) = (\rho A^{s-1}, \delta_\alpha)\ (\delta_\alpha A^{N-s}, \delta_N) \sim \lambda_1^{N-1} c_1 b^2 \qquad (8)$$

if $(\delta_\alpha, e_1) = b$.

The result follows from (6) - (8). ◆

Proof of E2(ii). Assume d = 1 for simplicity of notations. Let us fix some string $\xi(0) = \eta = (\eta_1,\dots, \eta_N)$ of length N. The distribution of s_n depends only on η_n. One can write

$$\sum_{t=1}^{\tau_n} f(\xi(t)) = \sum_{k=N}^{n} F_k$$

where

$$F_k = \sum_{t=\tau_k}^{\tau_{k-1}} f(\xi(t))$$

Note that $\frac{\tau_n}{N-n}$ converges to $Es = Es_n$ in probability as $s_n = \tau_{n-1} - \tau_n$ are conditionally (given η) independent random variables E here is the expectation with respect to ξ and after with respect to η). So one can consider

$$\frac{1}{N-n} \sum_{k=N}^{n} F_k$$

and the result follows. ■

Remark 1. Using more sophisticated results similar to random ergodic theorems (see [12]) one can prove a.s. convergence results.

Remark 2. One could also consider

$$\lim \frac{1}{N-n} \sum_{t=1}^{\tau_n} (n(\xi(t))-n(\xi(t-1))) = \lim \frac{1}{N-n} n(\xi(\tau_n)) = v.$$

In this case the sign of v does not depend on the choice of the stationary process $\eta(t)$.

5. Conjectures and problems.

1. One-dimensional generalised random walk (d is a fixed finite constant) can be null recurrent only on a set of Lebesgue measure 0 in the parameter space $\{ q(\gamma, \delta)\}$. One-dimensional generalised ergodic random walk (d is a fixed finite constant) always has the linear function $n(\alpha)$ as a Lyapounov function, satisfying

$$\sum_{\beta} p^{(k)}_{\alpha\beta} f(\beta) - f(\alpha) < -\varepsilon$$

for some $\varepsilon > 0$ and k which depend on the parameters. The same is true for transient cases.

2. Is it possible to calculate explicitly necessary and sufficient ergodicity conditions ?

3. Markov BCMP networks with LIFO discipline fall into the above introduced class of one-dimensional processes. FIFO disciplines lead to the class of processes with a localised interaction where both ends of a string are involved into the evolution. More exactly, a string $\gamma_1\alpha\gamma_2$ can become $\gamma_3\alpha\gamma_4$, $n(\gamma_i) \le d$, with some probability $q(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$

4. All stabilization laws hold for ergodic and transient one-dimensional chains. Which stabilization laws hold for null-recurrent one-dimensional chains ?

5. There are different approaches which can have some advantages in particular cases : using reversibility and(or) exlicit product form solutions, Kingman's subadditive ergodic theorem etc.

6. For the generalised two-dimensional random walks the results similar to those in section 2 seem possible to be proved.

To explain this more exactly define first N-dimensional generalised random walks. It is a discrete time homogeneous countable Markov chain $\xi(t)$ with the set of states

$$\mathcal{W}(N; k) = (\mathcal{W}(1; k))^N$$

so $\mathcal{W}(N; 1)$ can be identified with $\mathbb{Z}_+^N$ and an element of $\mathcal{W}(N; k) = \{ s = (s_{ij}, i = 1,..., k$, $j = 1,..., n_i = n_i(s)) \}$ consists of N strings of lengths n_i. We can think s_{ij} to be a customer type on the j-th place of the queue i, n_i being the length of the queue i. Its transition probabilities $p(s \to s')$ are non zero only if the following conditions are satisfied :

B : (boundedness of jumps) $| n_i(s') - n_i(s) | \le 1$;

H : (space homogeneity) Associated with the random process s(t) is the process $n(t) \in \mathbb{Z}_+^N$, where $n(t) = (n_1(t),..., n_N(t))$ is the queue length vector. We say that a state s belongs to a face B^Λ if n(s) does.Assume that $p(s \to s')$ depend on s only through $s_{i,n_i(s)}(t)$, $i = 1,..., k$, and through B^Λ to which s belongs; moreover $s'_{ij} = s_{ij}$ for $j \le n_i(s) - 1$. Similar to the one-dimensional case, we can introduce for the two-dimensional case the probabilities $q(\alpha,\beta; \gamma, \delta)$

of deleting one-character strings α, β from "the ends of queues" correspondingly and appending at most two-character strings γ and δ instead of them. They completely define transitions for $n_1(s), n_2(s) \geq 1$. On the axes we have also some probabilities $q(\emptyset, \alpha; \gamma, \delta)$ and $q(\alpha, \emptyset; \delta, \gamma)$ with $n(\alpha) = 1$, $n(\gamma) \leq 1$, $n(\delta) \leq 2$.

We would like to show that Fig.1 for the ordinary random walks holds also for the processes with a localised interaction.

Induced chain. The notion of the induced chain is much more complicated here comparatively with one customer case. There exist several possibilities for defining induced chains. We consider only one which we call a "rough induced chain". The idea behind it is that the absolute value of velocity is just a choice of scale and is of no importance.

We consider first the induced chain $\mathfrak{L}^{\{1,2\}}$. It should provide us with the behaviour of the process when the lengths of both strings are large.Consider a stationary process $\eta(t) = (\eta^1(t), \eta^2(t))$, $t \in \mathbb{Z}$, where both $\eta_i(t)$ have their values in $\{1,...,r\}$. Let the process $\xi^1(t)$, defined as in section 3 but with values in $\mathfrak{A}_1^2 = \{((k_1; x^1_{k_1}, x^1_{k_1+1},...), (k2; x^1_{k_2}, x^1_{k_2+1},...),\}$ starts from

$\xi^1(0) = \{((0; \eta_1(0), \eta_1(1), ...), (0; \eta_2(0), \eta_2(1),...))\}$

Conjecture 1. For almost all values of parameters there exist the limits

$$v_1 = \lim \frac{k_1(t)}{t}, \quad v_2 = \lim \frac{k_2(t)}{t}.$$

Moreover these limits are nonzero and sgn v_1, sgn v_2 do not depend on the choice of $\eta(t)$.

If sgn $v_2 > 0$ the face $B^{\{1\}}$ is called ergodic. Let us consider the case sgn $v_1 < 0$, sgn $v_2 > 0$ (corresponding to Fig. 2).

We shall define now the induced chain for the axis $B^{\{1\}}$. It is useful as one can visualise the deterministic motion along the axis $B^{\{1\}}$. This induced chain is the Markov chain $\xi(t)$ with the non countable state space $\mathfrak{A}_1 \times \mathcal{W}(1; k)$. It is useful to imagine two queues with two types of customers, then a state of the induced chain consists of the second queue together with a half-infinite first queue . Elements of this state space will be denoted by $(k; x_k, x_{k+1},...) \times \alpha$, where α is a finite string. Let us take again a stationary process $\eta(t)$ on $\{1,...,r\}^{\mathbb{Z}}$ and start $\xi(t)$ with $\xi(0) = (0; \eta(0), \eta(1),...) \times \emptyset$.

Conjecture 2. For almost all values of parameters there exist the limits, as $t \to \infty$,

$$\pi(\alpha) = \lim \mu_t(\mathfrak{A}_1 \times \alpha), \quad w = \lim \frac{k(t)}{t}, \quad \lim \mu_t(\{x_{r(t)} = \beta_0\} \times \{\alpha\}) = \pi(\alpha; \beta_0),$$

where μ_t is the time t distribution on the state space, k(t) is the value of k at time t. Moreover, w is non zero, sgn w does not depend on the choice of $\eta(t)$ and w can be defined by the formular similar to the formula (9) of section 3.

Remark 1. Other cases from Fig. 1 can be considered quite similarly and we get the classification in the same terms as for ordinary deflected random walks but in "less computable terms".

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