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C O N T E N T S

CLUSTER EXPANSION FOR UNBOUNDED NON-FINITE POTENTIAL

R. R. Akhmitzjanov, V. A. Malyshev and E. N. Petrova

Let $\mathbb{Z}^\nu \subset R^\nu$, $\nu \geq 2$, be the ν-dimensional lattice, $r_{tt'}$ the distance between $t,t' \in \mathbb{Z}^\nu$.

Let $\Lambda \subset \mathbb{Z}^\nu$ be a finite volume. We consider the Gibbs measure μ_Λ on $\mathbb{R}^\Lambda$:

$$\mu_\Lambda(x) = Z_\Lambda^{-1} \exp\{-\lambda \cdot U_\Lambda(x) - \sum_{t\in\Lambda} |x_t|^m\} dx \tag{1}$$

where $x=\{x_t, t\in\Lambda\} \in R^\Lambda$, $dx = \prod_{t\in\Lambda} dx_t$, $\lambda>0$ is small, $m>0$, and

$$U_\Lambda(x) = \sum_{t,t'\in\Lambda} U_{tt'}(x_t, x_{t'}) = \sum_{t,t'\in\Lambda} x_t^{2\kappa} x_{t'}^{2\kappa} r_{tt'}^{-(\nu+\varepsilon)} \tag{2}$$

$\kappa>0$ is an integer, $\varepsilon>0$, the sum runs over all pairs t,t' from Λ.

The partition function is

$$Z_\Lambda = \int_{R^\Lambda} \exp\{-\lambda U_\Lambda(x) - \sum_{t\in\Lambda} |x_t|^m\} dx .$$

Our main result is the following

Theorem. Let the parameters ν, m, ε, κ satisfy the inequality

$$m\nu + m\varepsilon - 2\nu\kappa \geq 0 . \tag{3}$$

Then there exists a $\lambda_0>0$ such that for each $0<\lambda<\lambda_0$ the partition function Z_Λ has the cluster expansion

$$Z_\Lambda = \sum_{\Gamma_1,\dots,\Gamma_n} C^{|\Lambda \setminus \cup\Gamma_i|} \kappa_{\Gamma_1} \cdot \dots \cdot \kappa_{\Gamma_n} \tag{4}$$

where the sum runs over all collections $\{\Gamma_1,\dots,\Gamma_n\}$ of pairwise nonintersecting subsets of Λ : $\Gamma_i \subset \Lambda$, $i=1,\dots,n$, C depends on the parameters λ and m , but is bounded from below by an absolute constant:

$$C = C(\lambda,m) \geq 2e^{-1} \tag{5}$$

We denote by $|A|$ the cardinality of the set $A \subset \mathbf{Z}^{\nu}$.

Moreover, the values of κ_Γ satisfy the cluster estimate: for each N

$$\sum_{\substack{\Gamma:\Gamma\ni o \\ |\Gamma|=N}} |\kappa_\Gamma| \quad (\delta(\lambda))^N \tag{6}$$

The sum in (6) runs over all sets $\Gamma \subset \mathbf{Z}^{\nu}$ such that Γ contains the origin and has fixed cardinality N , and $\delta(\lambda) \to 0$ when $\lambda \to 0$.

The potential $U_{tt'}$ in consideration is non-finite and unbounded. In the case of finite unbounded potential the only condition for the existence of the cluster expansions is the boundedness of the potential from below. For non-finite, but bounded from above potential condition $\varepsilon > 0$ is sufficient. Both these results were obtained in [2]. In both of these cases the initial independent measure is arbitrary and not necessarily has the density $\exp\{-\sum_{t\in\Lambda} |x_t|^m\}$ as we have in (1). Cluster expansion for non-finite unbounded potential is also established in [1]. In terms of our paper conditions in [1] are as follows: $m > 4\kappa$, $\varepsilon > 5\nu$. We improve these conditions.

The main idea of the expansion is the following. We choose some barrier B and in the case when the values of the random field in consideration are less than B we use the known techniques (see [2]) of expansion and estimation. We build some neighbourhoods of such $t \in \mathbf{Z}^{\nu}$, for which $|x_t| > B$, and unite them into clusters. In this case we get the cluster estimate because of the smallness of $\exp(-|x_t|^m)$.

Note that because of (4)

$$Z_\Lambda \cdot C^{-|\Lambda|} = \sum_{\Gamma_1,\dots,\Gamma_n} \left(\kappa_{\Gamma_1} C^{-|\Gamma_1|}\right)\dots\left(\kappa_{\Gamma_n} C^{-|\Gamma_n|}\right)$$

and hence the standard cluster techniques can be used only in the presence of an estimate:

$$\sum_{\substack{\Gamma:\Gamma\ni 0\\ |\Gamma|=N}} |\kappa_\Gamma| C^{-|\Gamma|} \le (\delta(\lambda))^N$$

but since we have (5) it is sufficient to prove (6).

Proof of the theorem. Cluster expansion

We need to formulate some definitions. We call a set $A \subset \mathbf{Z}^\nu$ connected iff for each $t,t' \in A$ there exists a sequence $t_1,\dots,t_n$ such that $t_i \in A$, $i=1,\dots,n$ and $r_{tt_1} = r_{t_1t_2} = \dots = r_{t_{n-1}t_n} = r_{t_nt'} = 1$.

We say that a collection of sets $\mathcal{T} = \{A_1,\dots,A_n\}$, $A_i \subset \mathbf{Z}^\nu$, $i=1,\dots,n$ is connected iff for each $\ell,m \in \{1,\dots,n\}$ there exists a sequence $i_1,\dots,i_\kappa$; $i_j \in \{1,\dots,n\}$, $j=1,\dots,\kappa$, such that $A_\ell \cap A_{i_1} \neq 0$, $A_{i_j} \cap A_{i_{j-1}} \neq \emptyset$ for all j and $A_{i_\kappa} \cap A_m \neq \emptyset$. We call $\Gamma = \bigcup_{i=1}^n A_i$ the support of the collection $\mathcal{T} = \{A_1,\dots,A_n\}$.

Now we shall describe the construction of the expansion (4). First we fix an arbitrary configuration $x = \{x_t, t\in\Lambda\}$ and construct clusters $\Gamma_1,\dots,\Gamma_n$ corresponding to the fixed configuration.

Let us put

$$B = B(\lambda) = \lambda^{-1/8\kappa} . \tag{7}$$

For each $t\in\Lambda$ with $|x_t|>B$ we construct the ν-dimensional neighbourhood O_t having the center t and radius R_t:

$$R_t = (|x_t|\cdot B^{-1})^{2\kappa/(\nu+\varepsilon)} . \tag{8}$$

Denote $M = \{t\in\Lambda: |x_t|>B\}$. Let $\mathcal{D}_1,\dots,\mathcal{D}_p$ be the maximal connected components of the set $\bigcup_{t\in M} O_t$. We shall refer to a $\mathcal{D}_i$ as a *drop*.

Let G be a graph with vertices $1,\ldots,p$ (note that p is the number of constructed drops), a line connecting i and j, $i\neq j$, exists iff there exist $t\in \mathcal{D}_i\cap M$ and $t'\in \mathcal{D}_j\cap M$ such that

$$\lambda^{1/2} x_t^{2\kappa} x_{t'}^{2\kappa} r_{tt'}^{-(\nu+\varepsilon)} \geq 1 . \tag{9}$$

In general G is not a connected graph. For each maximal connected component $\tilde{G}$ of G consider the union $\bigcup_{i\in\tilde{G}} \mathcal{D}_i$ with i running over all vertices of $\tilde{G}$. Changing the components $\tilde{G}$ we get the sets $A_1,\ldots,A_\ell$, $\bigcup_{i=1}^{\ell} A_i = \bigcup_{i=1}^{p} \mathcal{D}_i$. We will refer to $A_1,\ldots,A_\ell$ as *fragments*. So, the number of constructed fragments is equal to the number of connected components of G.

Let us denote by $T=T(x)$ the collection of such pairs (t,t'), that t and t' do not belong simultaneously to one and the same fragment. Note that for each $(t,t')\in T$

$$\lambda\cdot \mathcal{U}_{tt'}(x_t,x_{t'}) = \lambda x_t^{2\kappa} x_{t'}^{2\kappa} r_{tt'}^{-(\nu+\varepsilon)} \leq \sqrt{\lambda} . \tag{10}$$

In fact, if $|x_t|<B$ and $|x_{t'}|<B$ (10) follows from (7). If $|x_t|>B$ and $|x_{t'}|>B$, then since t and t' belong to different fragments, (9) is not fulfilled and hence (10) is true. If $|x_t|>B$ and $|x_{t'}|<B$ then since $t'\notin 0_t$ $r_{tt'}>R_t$, so

$$\lambda x_t^{2\kappa} x_{t'}^{2\kappa} r_{tt'}^{-(\nu+\varepsilon)} \leq \lambda\cdot B^{2\kappa} x_t^{2\kappa} R_t^{-(\nu+\varepsilon)} \leq \lambda\cdot B^{4\kappa} \leq \sqrt{\lambda} .$$

The following identity will be useful for us:

$$\exp\left\{-\lambda \sum_{(t,t')\in T} \mathcal{U}_{tt'}\right\} = \sum_{Q\subset T} \prod_{(t,t')\in Q} a_{tt'} \tag{11}$$

where the sum runs over all subsets $Q\subset T$ (including the empty set) and

$$a_{tt'} = \exp\{-\lambda \mathcal{U}_{tt'}\} - 1 \tag{12}$$

If $Q=\emptyset$ we put the corresponding term equal to 1.

We call each pair (t,t') a *link*. Let us fix an arbitrary $Q\subset T$. Let $\mathcal{T}$ be the collection of sets, consisting of all constructed fragments $A_1,\dots,A_\ell$ and all links belonging to Q. Let $T_1,\dots,T_n$ be the maximal connected subcollections of T, and respectively $\Gamma_1,\dots,\Gamma_n$ be their supports. We call each of $\Gamma_1,\dots,\Gamma_n$ a cluster, corresponding to fixed configuration x and $Q\subset T(x)$ and define

$$f_{T_i}(x) = \prod_{A\in T_i} \exp\{-\lambda U_A(x)\} \prod_{(t,t')\in T_i} a_{tt'} \prod_{t\in\Gamma_i} \exp\{-|x_t|^m\} \tag{13}$$

where the product $\prod_{A\in T_i}$ runs over all fragments belonging to T_i, $\prod_{(t,t')\in T_i}$ is meant over all links $(t,t')\in Q$ belonging to T_i, and

$$U_A(x) = \sum_{t,t'\in A} U_{tt'}(x_t,x_{t'}) = \sum_{t,t'\in A} x_t^{2\kappa}\, x_{t'}^{2\kappa}\, r_{tt'}^{-(\nu+\varepsilon)}$$

So, we have constructed a collection of clusters $\Gamma_1,\dots,\Gamma_n$ which corresponds to the fixed configuration x and fixed $Q\subset T(x)$, and defined the "weights" f_{T_i} of these clusters.

Consider an arbitrary collection $\Gamma_1,\dots,\Gamma_n$ of pairwise disjoint subsets $\Gamma_i\subset\Lambda$, $i=1,\dots,n$ (Γ_i is not necessarily connected). Let $X(\Gamma_1,\dots,\Gamma_n)\subset R^\Lambda$ be the set, consisting of configurations x with the following property: there exists $Q\subset T(x)$ such that $\Gamma_1,\dots,\Gamma_n$ is just the collection of clusters corresponding to (x,Q). Note that restriction of any $x\in X(\Gamma_1,\dots,\Gamma_n)$ on $\Lambda\setminus\bigcup_{i=1}^n\Gamma_i$ belongs to $[-B,B]^{\Lambda\setminus\bigcup_i\Gamma_i}$, hence, the set $X(\Gamma_1,\dots,\Gamma_n)$ can be represented as a direct product

$$X(\Gamma_1,\dots,\Gamma_n) = \left(\mathop{\times}_{i=1}^{n} X_{\Gamma_i}\right) \times [-B,B]^{\Lambda\setminus\bigcup_i\Gamma_i}$$

where X_{Γ_i} is exactly the set of configurations $x\in R^{\Gamma_i}$ such that for any $x\in X_{\Gamma_i}$ there exists $Q_i\subset T(x)$ such that the pair (x,Q_i) generates Γ_i.

For any finite $\Gamma\subset\mathbf{Z}^\nu$ denote

$$\kappa_\Gamma = \int_{X_\Gamma} \sum_{Q} f_T(x_\Gamma)\,dx_\Gamma \tag{14}$$

where $X_\Gamma \subset R^\Gamma$ is the set of such configurations x_Γ that there exists $Q \subset T(x_\Gamma)$ such that the pair (x_Γ, Q) generates just the cluster Γ, the sum $\sum_Q$ runs over all such Q, and $f_T(x_\Gamma)$ is defined in (13), $dx_\Gamma = \prod_{t\in\Gamma} dx_t$.

We assume $\kappa_\Gamma=0$ if $X_\Gamma=\emptyset$. Taking into account (14) we obtain the expansion (4) with

$$c = \int_{-B}^{B} \exp(-|y|^m)dy \geq \int_{-1}^{1} \exp(-|y|^m)dy \geq 2e^{-1} .$$

Indeed,

$$Z_\Lambda = \sum_{A_1,\ldots,A_\ell} \int_{\{x:A_1,\ldots,A_\ell\}} \prod_{i=1}^{\ell} \exp(-\lambda U_{A_i}) \exp\left(-\lambda \sum_{(t,t')\in T} U_{tt'}\right) \cdot$$

$$\cdot \exp\left(- \sum_{t\in\Lambda} |x_t|^m\right) dx .$$

The sum is meant over all collections of pairwise disjoint (not necessarily connected) sets $A_1,\ldots,A_\ell$, $A_i \subset \Lambda$, $i=1,\ldots,\ell$ and integration is over the set of configurations $x\in R^\Lambda$ such that $A_1,\ldots,A_\ell$ are exactly all fragments, generated by x, $T = (\Lambda\times\Lambda) \setminus \bigcup_{i=1}^{\ell} (A_i \times A_i)$.

Using now (11) for $\exp\left(-\lambda \sum_{(t,t')\ T} U_{tt'}\right)$ we obtain

$$Z_\Lambda = \sum_{A_1,\ldots,A_\ell} \sum_{Q\subset T} \int_{\{x:A_1,\ldots,A_\ell\}} \prod_{i=1}^{\ell} \exp(-\lambda U_{A_i}) \prod_{(t,t')\in Q} a_{tt'} \cdot$$

$$\cdot \exp\left(-\lambda \sum_{t\in\Lambda} |x_t|^m\right) dx .$$

Since a collection $A_1,\ldots,A_\ell$ together with Q determines uniquely the clusters $\{\Gamma_1,\ldots,\Gamma_n\}$, we may represent a summation as follows:

$$Z_\Lambda = \sum_{\Gamma_1,\ldots,\Gamma_n} \sum_{\{A_1,\ldots,A_\ell,Q\}} \int_{\{x:A_1,\ldots,A_\ell\}} \prod_{i=1}^{\ell} \exp(-\lambda U_{A_i}) \cdot$$

$$\cdot \prod_{(t,t')\in Q} a_{tt'} \exp\left(-\lambda \sum_{t\in\Lambda} |x_t|^m dx\right) .$$

Here the summation runs over all collections $\{\Gamma_1,\ldots,\Gamma_n\}$

of pairwise disjoint clusters $\Gamma_1,\ldots,\Gamma_n$, then over all fragments $A_1,\ldots,A_\ell$ and over all collections $Q\subset(\Lambda\times\Lambda)\setminus\bigcup_{i=1}^{\ell}(A_i\times A_i)$ of links, such that the collection of supports of maximal connected subcollections of the collection $\{A_1,\ldots,A_\ell , Q\}$ coincides with $\{\Gamma_1,\ldots,\Gamma_n\}$.

Performing now the integration over x_t , $t\in\Lambda\setminus\bigcup_{i=1}^{\ell}\Gamma_i$ and taking into account the definition of X_Γ , we get:

$$Z_\Lambda=\sum_{\Gamma_1,\ldots,\Gamma_n} C^{|\Lambda\setminus\bigcup_i\Gamma_i|}\prod_{i=1}^{n}\int_{X_{\Gamma_i}}\sum_{Q_i} f_{T_i}(x_{\Gamma_i})dx_{\Gamma_i} =$$

$$=\sum_{\Gamma_1,\ldots,\Gamma_n} C^{|\Lambda\setminus\bigcup_i\Gamma_i|}\kappa_{\Gamma_1}\cdots\kappa_{\Gamma_n} .$$

Proof of the theorem. Cluster estimate

First of all we obtain the cluster estimate for clusters, consisting only of fragments but not of links.

Let us fix the cardinality of Γ : $|\Gamma|=N$. We regard only the clusters Γ containing the origin: $\Gamma\ni 0$.

Moreover, let us assume at first that Γ is a ν-dimensional spherical neighbourhood of some t . Then, obviously, $|x_t|>B$. We will show now that

$$\exp(-\tfrac{1}{3}|x_t|^m)\leq(\sqrt{\lambda})^N . \tag{15}$$

Let 0 be a ν-dimensional sphere having radius R , and N - the number of points in 0 : $N=|\mathbf{Z}^\nu\cap 0|$. There exist some constants c_1 and c_2 (depending on ν) such that

$$c_2R^\nu\leq N\leq c_1R^\nu \tag{16}$$

and therefore it is sufficient to show that

$$\exp(-\tfrac{1}{3}|x_t|^m)\leq(\sqrt{\lambda})^{c_1R_t^\nu}$$

or, taking the logarithm and using (8)

$$|x_t|^m\geq\frac{3}{2}c_1(x_t\cdot B^{-1})^{2\kappa\nu/(\nu+\varepsilon)}\cdot(-\ln\lambda) .$$

Together with (7) it leads to

$$|x_t|^{\frac{m(\nu+\varepsilon)-2\kappa\nu}{(\nu+\varepsilon)}} \geq \frac{3}{2}\, c_1\, \lambda^{\nu/4(\nu+\varepsilon)}\cdot(-\ln\lambda)\ .$$

If (3) holds and λ is sufficiently small, the last inequality holds, too. Consequently, if Γ is a sphere and $|\Gamma|=N$, then

$$|\kappa_\Gamma| \leq (\sqrt{\lambda})^N \int \exp\left\{-\frac{1}{3}\sum_{t\in\Gamma}|x_t|^m\right\} \prod_t dx_t \leq (C\sqrt{\lambda})^N \tag{17}$$

where the constant C depends on m .

Now, let Γ be a drop, i.e. $\Gamma = \bigcup_t 0_t$ where 0_t is a spherical neighbourhood of t . Note that Γ is connected in this case. Let us choose $M\subset\Gamma$ a subset of the centers of spherical neighbourhoods for which $\Gamma = \bigcup_{t\in M} 0_t$. (It can be done in at most 2^N different ways.) For any $t\in M$ similarly to (15) we have

$$\exp(-\frac{1}{3}|x_t|^m) \leq (\sqrt{\lambda})^{|0_t|}$$

and since $\sum_{t\in M}|0_t| \geq |\Gamma| = N$ we have (17) with another constant C . (C denotes different constants not depending on λ .)

Note that Γ is connected, $|\Gamma|=N$ is fixed and $\Gamma\ni 0$. The number of different sets $\Gamma\subset\mathbb{Z}^\nu$ with these properties does not exceed C^N for some constant C , depending on ν . Therefore, taking into account (17), we get (6) for connected Γ .

Now we are going to prove the cluster estimate in the case when Γ is not connected. Then, according to our construction, if Γ contains no links it consists only of one fragment. We regard at first fragments containing only two drops, i.e. Γ is the union of two connected sets. Let us denote these drogs by $\mathcal{D}_1$ and $\mathcal{D}_2$ and let $\mathcal{D}_1\ni 0$. Let us fix $\mathcal{D}_1$ and let $|\mathcal{D}_1|=N_1$. We fix also $t_1\in\mathcal{D}_1$ such that

$$|x_{t_1}| = \max_{t\in\mathcal{D}_1} |x_t| = b_1\ .$$

Since the drops $\mathcal{D}_1$ and $\mathcal{D}_2$ form one fragment, there exists $t_2 \in \mathcal{D}_2$ such that

$$\sqrt{\lambda}\ b_1^{2\kappa} b_2^{2\kappa}\ r_{t_1 t_2}^{-(\nu+\varepsilon)} \geq 1$$

where $b_2 = |x_{t_2}|$. That is, if t_1, b_1 and b_2 are fixed, t_2 belongs to the spherical neighbourhood of t_1 having radius $(b_1 b_2)^{2\kappa/(\nu+\varepsilon)}$, so that the number of different ways of fixing t_2 does not exceed $c_1 (b_1 b_2)^{2\kappa\nu/(\nu+\varepsilon)}$. Denoting $N_2 = N - N_1$ and summing through all connected $\mathcal{D}_2$, having cardinality N_2 and containing fixed t_2 , similarly to (17) we have:

$$\sum_{\substack{\mathcal{D}_2 : \mathcal{D}_2 \ni t_2 \\ |\mathcal{D}_2| = N_2}} \int \exp(-\lambda U_{\mathcal{D}_2}) \exp\left(-\sum_{t \in \mathcal{D}_2} |x_t|^m\right) \prod_{t \in \mathcal{D}_2 \setminus t_2} dx_t \leq$$
$$\leq \exp(-\tfrac{2}{3}\, b_2^m) \cdot (C\sqrt{\lambda})^{N_2} . \tag{18}$$

The left hand side integral is over the set of such configurations on $\mathcal{D}_2$, that $\mathcal{D}_2$ is a drop and $x_{t_2} \equiv b_2$ is fixed.

Similarly,

$$\sum_{\substack{\mathcal{D}_1 : \mathcal{D}_1 \ni 0 \\ |\mathcal{D}_1| = N_1}} \int \exp\ (-\lambda U_{\mathcal{D}_1})\ \exp\left(-\sum_{t \in \mathcal{D}_1} |x_t|^m\right) \prod_{t \in \mathcal{D}_1 \setminus t_1} dx_t \leq$$
$$\leq \exp(-\tfrac{2}{3}\, b_1^m) \cdot (C\sqrt{\lambda})^{N_1}$$

where the sum runs over all connected $\mathcal{D}_1$ having cardinality N_1 and containing the origin, integration is over the set of such configurations on $\mathcal{D}_1$ that $\mathcal{D}_1$ is a drop and $x_{t_1} \equiv b_1$ is fixed.

Consequently,

$$\sum_{\substack{\Gamma:\Gamma\ni 0\\ |\Gamma|=N}} |\kappa_\Gamma| \le \int db_1 db_2 \sum_{\mathcal{D}_1\ni 0}\ \sum_{t_1\in\mathcal{D}_1} \Big\{\int \exp(-\lambda U_{\mathcal{D}_1}) \cdot$$

$$\cdot \exp\Big(-\sum_{t\in\mathcal{D}_1}|x_t|^m\Big)\prod_{t\in\mathcal{D}_1\setminus t_1} dx_t\Big\} \sum_{t_2}\ \sum_{\substack{\mathcal{D}_2\ni t_2\\ |\mathcal{D}_2|=N-|\mathcal{D}_1|}} \Big\{\int \exp(-\lambda U_{\mathcal{D}_2})\cdot$$

$$\cdot \exp\Big(-\sum_{t\in\mathcal{D}_2}|x_t|^m\Big)\prod_{t\in\mathcal{D}_2\setminus t_2} dx_t\Big\} .$$

Substituting (18) and (19) into the last inequality we obtain

$$\sum_{\substack{\Gamma:\Gamma\ni 0\\ |\Gamma|=N}} |\kappa_\Gamma| \le (C\sqrt{\lambda})^N \int_B^\infty b_1^{2\kappa\nu/(\nu+\varepsilon)} \exp(-\tfrac{2}{3}b_1^m) \cdot$$

$$\cdot \int_B^\infty b_2^{2\kappa\nu/(\nu+\varepsilon)} \exp(-\tfrac{2}{3}b_2^m)\, db_2 \le (C\sqrt{\lambda})^N .$$

Now we shall examine the case when a fragment contains an arbitrary number s of drops. Let us fix $\mathcal{D}_1\ni 0$ and $t_1\in\mathcal{D}_1$. For each fragment there exists a tree $\mathcal{Y}$ with vertices $t_1,t_2,\dots,t_s$, $t_i\in\mathbb{Z}^\nu$ such that $t_i\in\mathcal{D}_i$ and a line between t_i and t_j meaning that

$$\sqrt{\lambda}\, x_{t_i}^{2\kappa}\, x_{t_j}^{2\kappa}\, r_{t_i t_j}^{-(\nu+\varepsilon)} \ge 1 . \qquad (20)$$

Let us denote $b_i=x_{t_i}$, $i=1,\dots,s$. Let us fix $t_2,\dots,t_s$, $b_1,\dots,b_s$, $b_i\ge B$ for each i , and a tree $\mathcal{Y}$, satisfying (20). Let us fix integers $N_1,\dots,N_s$, such that $\sum N_i = N$ (i.e. we fix the cardinalities of the drops $\mathcal{D}_1,\dots,\mathcal{D}_s$, of a fragment). Obviously,

$$\exp(-\lambda U_\Gamma)\exp\Big(-\sum_{t\in\Gamma}|x_t|^m\Big) \le \prod_{i=1}^{s}\Big\{\exp(-\lambda U_{\mathcal{D}_i})\cdot\exp\Big(-\sum_{t\in\mathcal{D}_i}|x_t|^m\Big)\Big\}. \qquad (21)$$

Using (18) for each $i=2,\dots,s$ we have

$$\sum_{\substack{\mathcal{D}_i:\mathcal{D}_i\ni t_i\\ |\mathcal{D}_i|=N_i}} \int \exp(-\lambda U_{\mathcal{D}_i})\cdot\exp\left(-\sum_{t\in\mathcal{D}_i}|x_t|^m\right)\prod_{t\in\mathcal{D}_i\setminus t_i} dx_t \le \tag{22}$$

$$\le \exp(-\tfrac{2}{3}\, b_i^m)\,(C\sqrt{\lambda})^{N_i} .$$

Let us keep $b_1,\dots,b_s$ fixed and estimate the number of trees $\mathcal{Y}$, containing a fixed vertex t_1 and satisfying (20). We shall describe an algorithm, which enumerates all such trees $\mathcal{Y}$.

An algorithm, enumerating the trees

1. step. Fix a vector of nonnegative integers $(n_1,\dots,n_s)$ such that $n_1\neq 0$, $n_s=0$ and $\sum n_i=s-1$. Evidently, it can be done in at most 4^s different ways.

2. step. Choose n_1 vectors $v_1,\dots,v_{n_1}$, $v_i\in\mathbf{Z}^\nu$, $i=1,\dots,n_1$ satisfying the following conditions: v_i belongs to the spherical neighbourhood of the origin having radius $(b_1 b_{i+1})^{2\kappa/(\nu+\varepsilon)}$. Construct:

$$\begin{aligned} t_2 &= t_1 + v_1\\ t_3 &= t_1 + v_2\\ &\dots\dots\\ t_{n_1+1} &= t_1 + v_{n_1} . \end{aligned}$$

Construct the lines between t_1 and each of the $t_2,\dots,t_{n_1+1}$.

3. step. Choose the first (in lexicographic order) of the constructed vertices, excluding t_1. Let this vertex be t_j. If $n_2\neq 0$, choose n_2 vectors $v_{n_1+1},\dots,v_{n_1+n_2}$, $v_i\in\mathbf{Z}^\nu$ for each i, such that v_i belongs to the spherical neighbourhood of the origin having radius $(b_j b_{i+1})^{2\kappa/(\nu+\varepsilon)}$. Construct

$$\begin{aligned} t_{n_1+2} &= t_j + v_{n_1+1}\\ &\dots\dots\dots\\ t_{n_1+n_2+1} &= t_j + v_{n_1+n_2} . \end{aligned}$$

Construct the lines between t_j and each of

$t_{n_1+2}, \dots, t_{n_1+n_2+1}$. If $n_1=0$, pass to the next step without any construction.

We proceed by induction. Let p steps be already performed and $n_1+n_2+\dots+n_{p-1}+1$ vertices be constructed.

(p+1). step. Choose the first (in lexicographic order) of all vertices having been constructed during the previous steps excluding those which have been already chosen earlier. Let this vertex be t_ℓ . If $n_p=0$, pass to the next step without any construction.

If $n_p \neq 0$, choose n_p vectors $v_{n_1+\dots+n_p+1}, \dots, v_{n_1+\dots+n_p}$, $v_i \in \mathbb{Z}^\nu$ for each i , such that v_i belongs to the spherical neighbourhood of the origin having radius $(b_\ell b_{i+1})^{2\kappa/(\nu+\varepsilon)}$.

Construct

$$t_{n_1+\dots+n_{p-1}+2} = t_\ell + v_{n_1+\dots+n_{p-1}+1}$$

$$\dots\dots\dots\dots\dots\dots$$

$$t_{n_1+\dots+n_p+1} = t_\ell + v_{n_1+\dots+n_p} .$$

Construct the lines between t_ℓ and each of the vertices constructed in this step.

After s steps the construction is finished.

Thus, if $b_1, \dots, b_s$, $n_1, \dots, n_{s-1}$ and $v_1, \dots, v_{s-1}$ are chosen, the graph constructed by our algorithm is unique. Choosing all possible $b_1, \dots, b_s$, $n_1, \dots, n_{s-1}$, $v_1, \dots, v_{s-1}$ we can construct among other graphs all possible trees $\mathcal{Y}$. Moreover, we construct each tree more times. Indeed, let T be a tree having s vertices and a root t_1 . Let us denote by n_1 the number of lines in T incident with t_1 , (n_i+1) - the number of lines in T incident with t_i , $i=2,\dots,s$. Then the tree T is constructed by our algorithm at last $\prod_{i=1}^{s} (n_i!)$ times.

In fact, let $\tilde{b}_1, \dots, \tilde{b}_s$ and $\tilde{v}_1, \dots, \tilde{v}_{s-1}$ be the vectors, generating the tree T . Let us divide a collection $(\tilde{b}_1, \dots, \tilde{b}_s)$ into subcollections in the following way. The first subcollection consists of only one element, namely

$\tilde{b}_1$. The second subcollection consists of n_1 vectors: $\tilde{b}_2,\ldots,\tilde{b}_{n_1+1}$; the third one consists of the next n_2 vectors, and so on. Consider now a new collection $(\bar{b}_1,\ldots,\bar{b}_s)$, obtained from $(\tilde{b}_1,\ldots,\tilde{b}_s)$ by arbitrary permutations in each of the described subcollections (but without any permutations between the subcollections). Let $\bar{v}_1,\ldots,\bar{v}_{s-1}$ be a collection, obtained from $\tilde{v}_1,\ldots,\tilde{v}_{s-1}$ by the same permutations. Obviously, the tree generated by $\bar{b}_1,\ldots,\bar{b}_s,\ \bar{v}_1,\ldots,\bar{v}_{s-1}$ is T , and there are exactly $\Pi(n_i!)$ such permutations.

The number of different ways of choosing $v_1,\ldots,v_{s-1}$, having $b_1,\ldots,b_s$ and $n_1,\ldots,n_{s-1}$ fixed, is

$$C_1^s\, b_1^{2\kappa n_1 \nu/(\nu+\varepsilon)} \prod_{j=2}^{s} b_{i_j}^{2\kappa(n_j+1)/(\nu+\varepsilon)} \tag{23}$$

where we have denoted by i_j the number of the vertex, which is chosen as the first one in lexicographic order in the $(j+1)^{th}$ step of the algorithm, $j=2,\ldots,s-1$, and used (16).

Note that

$$\sup_{b_i} b_i^{2\kappa(n_i+1)/(\nu+\varepsilon)} \exp(-\tfrac{1}{3}b_i^m) \le (Cn_i)^{n_i} . \tag{24}$$

Taking into account (21), (22), (23) and (24) finally we have:

$$\sum_{\substack{\Gamma:\Gamma\ni 0\\ |\Gamma|=N}} |\kappa_\Gamma| \le (C\sqrt{\lambda})^N \sum_{s\le N} \sum_{(n_1,\ldots,n_{s-1})} \frac{1}{(\Pi n_i!)} \prod_{i=1}^{s} \Big[(Cn_i)^{n_i} \cdot \int_B^\infty \exp(-\tfrac{1}{3}b_i^m)\,db_i \Big] \le (C\sqrt{\lambda})^N .$$

Now we must consider the case when Γ contains links. Since

$$\sum_{t'\in\mathbb{Z}^\nu} |a_{tt'}| \le \sum_{t'\in\mathbb{Z}^\nu} \lambda B^{4\kappa} r_{tt'}^{-(\nu+\varepsilon)} \le C\sqrt{\lambda} \tag{26}$$

the case when Γ consists only of links is trivial. The general case follows from (26) and (25) by induction on the number of links and fragments in the cluster.

References

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