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Markov Processes and Related Fields

Two-Sided Evolution of a Random Chain

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Abstract. A finite chain (string) is just a sequence of symbols from some finite alphabet. We consider Markov chains with the state space equal to the set of all finite strings. In the simplest situation left-sided evolution of the string is defined by the following one-step transition probabilities: $q_l(x, \emptyset)$ is the probability that the leftmost symbol of the string is erased, if this equals x; $q_l(x, y)$ is the probability that the leftmost symbol x is substituted by y; $q_l(x, yz)$ is the probability that the leftmost symbol x is substituted by yz.

Right-sided evolution is defined similarly. We consider the case when left and right evolution occur simultaneously and independently. In the generic situation we obtain a complete classification of such Markov chains.

Keywords: Markov chain, string, invariant measure, stabilisation law

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1. Introduction

This paper continues the study of Markov chains that govern the evolution of finite and semi-infinite sequences of symbols (see [2–5]), but it can be read independently.

Here we would like to explain the main result without being completely rigorous. More precise definitions and technical conditions will be given in the next section and the reader can consult it while reading this introduction.

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A finite chain (string) is just a sequence of symbols from a finite alphabet $S = \{1, 2, \ldots, r\}$. We consider Markov chains with the state space equal to the set of all finite strings. We consider three different types of evolution: one-sided (left and right) and two-sided. Left-sided evolution of the string is defined by the following one-step transition probabilities:

- $q_l(x, \emptyset)$ denotes the probability that the leftmost symbol of the string is erased, if it is equal to x;
- $q_l(x, y)$ is the probability that the leftmost symbol x is substituted by y;
- $q_l(x, yz)$ is the probability that the leftmost symbol x is substituted by yz.

Of course we assume for all x that

$$q_l(x, \emptyset) + \sum_y q_l(x, y) + \sum_{yz} q_l(x, yz) = 1.$$

The transitions given here depend only on the last symbol. Moreover the lengths of the string at subsequent moments of time cannot differ by more than 1, but in the paper we will study a more general evolution. The above parameters do not define the evolution when the string is empty (otherwise speaking, when its length equals zero), but we simply assume that the jumps from the empty string are somehow defined and can only occur to strings of length one (non-degeneracy conditions are important here).

The evolution is called transient if the length $n(t)$ of the string tends to infinity a.s. One can then show the following:

- if $t \to \infty$ then
$$\frac{n(t)}{t} \to v_l$$
for some constant v_l;
- when $n(t)$ becomes large, the distribution of the symbols inside the string, but not too close to its ends, tends to those of some stationary random process with the exponential mixing property. Denote by μ_l this limiting distribution.

We consider also left-semi-infinite strings, i.e. sequences $\alpha = x_1 x_2 \ldots$ with values in the same alphabet. Left-hand evolution is defined similarly by the same parameters q_l.

It is more convenient however to view the semi-infinite string as a function $x(i, t)$ on $\mathbf{Z}$ with values in $\mathbf{Z} \cup \{0\}$, where 0 corresponds to the "vacuum". At time t this function is not equal to 0 on some interval $[a_t, \infty]$ and it is equal to 0 outside this interval. Here a_t denotes the position of the leftmost symbol of the string. At time 0 the string $\alpha = x_1 x_2 \ldots$ is identified with the function equal to x_i for $i > 0$, and to the vacuum for $i \leq 0$. If $x(a_t, t) = x$ then, for example, with probability $q(x, yz)$ the next state will be given by $x(a_t, t+1) = z$, $x(a_t - 1, t+1) = y$ and all other values remain the same. Thus $a_{t+1} = a_t - 1$ in this case.

We always take the initial distribution μ of semi-infinite strings as the restriction to $\mathbf{Z}_+$ of some stationary random process on $\mathbf{Z}$. If the corresponding Markov chain on finite strings is ergodic then $a_t \to \infty$ a.s., so that

$$\frac{a_t}{t} \to v_{l,\mathrm{erg}}(\mu).$$

In a similar way we can define right evolution and we denote the corresponding parameters by

$$q_r,\ v_r,\ \mu_r,\ v_{r,\mathrm{erg}}(\mu)$$

$$\frac{b_t}{t} \to -v_{r,\mathrm{erg}}(\mu),$$

where b_t is the coordinate of the right end.

Two-sided evolution of finite strings is defined by independent evolution of the left and right ends with corresponding parameters q_l, q_r. The three parameters $v_l, \mu_l, v_{r,\mathrm{erg}}(\mu_l)$ plus their couterparts play a fundamental role in the classification of two-sided evolution. The most difficult case is when one string, the left one say, is transient and the right string is ergodic. Our main result is that in the two-sided evolution the length of the string tends to infinity a.s. if

$$v_l > v_{r,\mathrm{erg}}(\mu_l)$$

and its mean length stays bounded if

$$v_l < v_{r,\mathrm{erg}}(\mu_l).$$

In the next section we will formulate all these and a number of other definitions and results in a precise way. The other sections contain the proofs of these results. In the last section we will give an example.

2. Definitions and Main Results

2.1. Two-sided finite strings

Fix a finite set (an alphabet)

$$\mathcal{S} = \{1, \dots, r\}$$

consisting of r symbols. A finite string is a finite sequence of symbols from $\mathcal{S}$:

$$\alpha = x_1 \dots x_n, \quad x_i \in \mathcal{S}.$$

We denote by $|\alpha| = n$ the length of the sequence α, and by $\emptyset$ the empty string of length 0.

For two arbitrary sequences $\alpha = x_1 \dots x_n$ and $\beta = y_1 \dots y_m$ we define their concatenation (of length $m+n$) by

$$\alpha\beta = x_1 \dots x_n y_1 \dots y_m.$$

Let

$$\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{S}^n$$

be the set of all finite sequences. For any finite sequence γ let $\mathcal{A}\ (\gamma)$ be the set of all sequences $\alpha = \gamma\rho, \rho \in \mathcal{A}$, and $\mathcal{A}_k$ the set of all sequences of length less than k.

Consider a discrete time homogeneous countable Markov chain $\mathcal{L}_0$ on the set $\mathcal{A}$. Let $\xi(t)$ be the state of the Markov chain $\mathcal{L}_0$ at time t.

We assume that the one-step transition probabilities $p_{\alpha,\tilde{\alpha}}$ satisfy the following conditions. Fix some natural number d.

Condition 2.1. (*Spatial homogeneity*). *For $|\alpha| \geq 2d$, the transition probabilities $p_{\alpha,\tilde{\alpha}} \neq 0$ only if $\alpha = \gamma_l\rho\gamma_r, \tilde{\alpha} = \theta_l\rho\theta_r$ for some $\gamma_l, \gamma_r, \theta_l, \theta_r, \rho$ with $|\gamma_l| = |\gamma_r| = d$, $|\theta_l|, |\theta_r| \leq 2d$. Moreover, the transition probabilities $p_{\alpha,\tilde{\alpha}} \equiv p_{\gamma_l\rho\gamma_r,\theta_l\rho\theta_r}$ do not depend on ρ but only on $\gamma_l, \gamma_r, \theta_l, \theta_r$. By definition we put*

$$q(\gamma_l, \theta_l, \gamma_r, \theta_r) = p_{\gamma_l\rho\gamma_r,\theta_l\rho,\theta_r}. \tag{2.1}$$

Condition 2.2. (*Independence*). *For $|\alpha| \geq 2d$,*

$$q(\gamma_l, \theta_l, \gamma_r, \theta_r) = q_l(\gamma_l, \theta_l) q_r(\gamma_r, \theta_r)$$

for some parameters $q_l(\gamma, \theta), q_r(\gamma, \theta) \geq 0$ such, that for all γ with $|\gamma| = d$

$$\sum_{\theta:|\theta|\leq 2d} q_l(\gamma, \theta) = 1, \qquad \sum_{\theta:|\theta|\leq 2d} q_r(\gamma, \theta) = 1.$$

So the ends are independent of each other. We delete the substring γ_l from the left, and instead we append the substring θ_l with probability $q_l(\gamma_l, \theta_l)$. We change the right end with probability $q_r(\gamma_r, \theta_r)$ in an analogous manner.

Condition 2.3. (*Non-degeneracy*). *Suppose that all probabilities*

$$q_l(\gamma, \theta) \neq 0, \ \ q_r(\gamma, \theta) \neq 0$$

for all strings γ, θ such that $|\gamma| = d, |\theta| \leq 2d$. Assume also positivity of all transition probabilities $p_{\alpha,\tilde{\alpha}}$ for $|\alpha| < 2d$, $|\tilde{\alpha}| = 2d$ and assume $p_{\alpha,\tilde{\alpha}} = 0$ for $|\alpha| < 2d$, $|\tilde{\alpha}| - |\alpha| > d$.

These conditions imply that all states are essential and that the Markov chain $\mathcal{L}_0$ is irreducible and aperiodic. The central assumptions are homogeneity of the transition probabilities for all strings of length greater than $2d$ and independence of the ends. Condition 2.3 is assumed only to simplify formulation.

Together with the above defined countable Markov chain $\mathcal{L}_0$, the states of which are finite strings, we consider a Markov process on marked strings. Define a marked string as a triplet (α, a, b), where $a \le b,\ a, b \in \mathbf{Z}$ and $\alpha = x_a, \dots, x_b$ is a finite sequence of symbols from $\mathcal{S}$. Denote by $\mathcal{C}$ the set of all marked strings. By $\alpha_{[c,d]}$ we denote the substring $\alpha_{[c,d]} = x_c \dots x_d,\ a \le c \le d \le b$.

Next define the following countable Markov chain $\mathcal{L}$ on the set $\mathcal{C}$. Let $\xi(t)$ be the state of $\mathcal{L}_0$ at time t. Then the state of $\mathcal{L}$ at time t is the triplet $(\xi(t), a_t, b_t)$, where a_t, b_t are defined in the following way. Let $a_0 \in \mathbf{Z},\ b_0 = a_0 + |\xi(0)|$. If $|\xi(t)| \ge 2d$, $\xi(t) = \gamma_l \rho \gamma_r$, where $|\gamma_l| = |\gamma_r| = d$ and $\xi(t+1) = \theta_l \rho \theta_r$, then define

$$\begin{aligned} a_{t+1} &= a_t + |\gamma_l| - |\theta_l|, \\ b_{t+1} &= b_t + |\theta_r| - |\gamma_r|. \end{aligned}$$

If $|\xi(t)| < 2d$, then (somewhat arbitrarily) put

$$\begin{aligned} a_{t+1} &= a_t, \\ b_{t+1} &= a_{t+1} + |\xi(t+1)|. \end{aligned}$$

Definition 2.1. We say that $\mathcal{L}$ is ergodic if $\mathcal{L}_0$ is ergodic.

2.2. Semi-infinite strings

Together with the above defined countable Markov chain $\mathcal{L}$, the states of which are finite strings, we will consider a Markov chain on semi-infinite strings. Define a semi-infinite string as a pair (α, a), where $a \in \mathbf{Z}$ and α is an infinite sequence of symbols from $\mathcal{S}$

$$\alpha = x_1 x_2 \dots x_k \dots, \quad x_i \in \mathcal{S}.$$

Let

$$\mathcal{S}^{[a,\infty]} = \{\alpha : \alpha = x_a x_{a+1} x_{a+2} \dots,\ x_i \in \mathcal{S}\}.$$

Define the Markov chain $\mathcal{L}_\infty$ on the set $\bigcup_{a \in \mathrm{Z}} \left(\mathcal{S}^{[a,\infty)}, \{a\}\right)$ with the following dynamics. If at time t the state of the chain $\mathcal{L}_\infty$ is $(\eta(t), a_t)$ and $\eta(t) = \gamma\rho$, $|\gamma| = d$, then with probability $q_l(\gamma, \theta)$ the state at time $t+1$ will be $\eta(t+1) = \theta\rho, a_{t+1} = a_t + |\gamma| - |\theta|$.

We shall denote by

$$\mathcal{B}_\infty = \mathcal{S}^{[1,\infty)}$$

the set of all semi-infinite sequences. We suppose $\mathcal{B}_\infty$ to be equipped with the product topology. So $\mathcal{B}_\infty$ is a compact, metrisable space.

Let $\mathcal{P}_\infty$ be the set of all probability measures on $\mathcal{B}_\infty$. We denote by $p_\varphi(\gamma)$, $\gamma \in \mathcal{A}$, the correlation functions (or the finite-dimensional distributions) corresponding to the measure φ:

$$p_\varphi(\gamma) = \varphi(\{\alpha \in \mathcal{B}_\infty : \alpha_{[1,|\gamma|]} = \gamma\}), \tag{2.2}$$

for all $\gamma \in \mathcal{A}$, where $\alpha_{[1,|\gamma|]}$ is the sequence $x_1 \dots x_{|\gamma|}$, if $\alpha = x_1, x_2 \dots$

Let

$$(\eta(t), a_t) = (\eta_1(t)\eta_2(t)\dots\eta_k(t)\dots, a_t), \quad \eta_k(t) \in \mathcal{S},$$

be the state of $\mathcal{L}_\infty$ at time t and $\varphi P(t) \in \mathcal{P}_\infty$ be the distribution of $\eta(t)$ under the condition that the initial state has distribution (φ, δ_0), $\varphi \in \mathcal{P}_\infty$, i.e. a is fixed to 0 ($a_0 = 0$). We denote by $p_t(\gamma \mid \varphi)$ the correlation functions of $\eta(t)$, given that the initial state has distribution (φ, δ_0).

Definition 2.2. We say that a measure $\varphi \in \mathcal{P}_\infty$ is invariant with respect to Markov chain $\mathcal{L}_\infty$ if $\varphi P(t) = \varphi$.

Let us denote by $\mathcal{T}$ the set of all translation invariant measures on $\mathcal{B}_\infty$, in other words a measure belongs to $\mathcal{T}$ if it is the restriction of some translation invariant measure on $\mathcal{S}^{\mathrm{Z}}$. Furthermore, denote by $\mathcal{E} \subset \mathcal{T}$ the subset of all ergodic measures on $\mathcal{B}_\infty$ amongst these.

We can define a process $\mathcal{L}_{-\infty}$ on the set $\bigcup_{b \in \mathrm{Z}} \mathcal{S}^{(-\infty,b]}$ with left end $a = -\infty$ and the corresponding sets $\mathcal{B}_{-\infty}$ and $P_{-\infty}$ in the same way as before.

2.3. One-sided strings

Suppose that $q_r(\gamma, \gamma) = 1$. Then only the left end of the marked string can change. Let the initial position of the right end be -1; this does not change over time. The state of the process at time t is hence $(\xi(t), -|\xi(t)|)$, because the position of the left end is defined by the length of $\xi(t)$.

This Markov chain will be denoted by $\mathcal{L}_l$. In the same way we can assume that only the right end can change. Denote this Markov chain by $\mathcal{L}_r$.

In the sequel we will also say that the state of Markov chain $\mathcal{L}_l$ is $\xi(t)$, since the other parameters of the string are determined by $\xi(t)$.

Remark 2.1. For the Markov chain $\mathcal{L}_l$ ($\mathcal{L}_r$) we assume the following condition to hold.

Condition 2.4. (*Non-degeneracy*). $q_l(\gamma, \theta) \neq 0$, ($q_r(\gamma, \theta) \neq 0$) *for all strings* γ, θ *such that* $|\gamma| = d$, $|\theta| \leq 2d$.

2.4. Transient strings

2.4.1. Stabilisation laws for finite strings

Let the Markov chain $\mathcal{L}_l$ be transient. For all $\alpha \in \mathcal{A}, \alpha \neq \emptyset$, define the following correlation functions:

$$p_t(\alpha \mid \beta) = \sum_{\rho \in \mathcal{A}} \mathsf{P}\{\xi(t) = \alpha\rho \mid \xi(0) = \beta\}. \tag{2.3}$$

It is the probability that the left end of string $\xi(t)$ is equal to α at time t under the condition that the initial state is β. Our aim is to study the long-time behaviour of these correlation functions.

The following theorem has been proved in [3].

Theorem 2.1. *Let Conditions* 2.1, 2.4 *hold and let* $\mathcal{L}_l$ *be transient. Then the following assertions hold:*
(i) *For all* $\alpha\in\mathcal{A}$, $\alpha\neq\emptyset$ *and for all initial states* $\beta\in\mathcal{A}$

$$\lim_{t\to\infty} p_t(\alpha\mid\beta)=p_\mu(\alpha), \tag{2.4}$$

where $p_\mu(\alpha)$, $\alpha\in\mathcal{A}$, *are the correlation functions of some measure* $\mu\in\mathcal{P}_\infty$. *Moreover, the convergence in* (2.4) *is exponentially fast, that is, there is some* $\chi>0$ *such that*

$$|p_t(\alpha\mid\beta)-p_\mu(\alpha)|\leq C(|\alpha|)e^{-\chi t}, \tag{2.5}$$

for some constant $C(|\alpha|)$ *depending only on* $|\alpha|$.
(ii) *For any initial state*

$$\lim_{t\to\infty}\frac{|\xi(t)|}{t}=v_l(\mu)>0 \tag{2.6}$$

in probability, where the "velocity" $v_l(\mu)$ *is given by the formula*

$$v_l(\mu)=\sum_{\gamma:|\gamma|=d}p_\mu(\gamma)\sum_{\theta:|\theta|\leq 2d}(|\theta|-d)\,q_l(\gamma,\theta). \tag{2.7}$$

Next we give a formula for the correlation functions $p_\mu(\alpha)$ of the measure μ; another formula was given in [2]. Introduce the following notation. For all θ,γ and θ_1 such that $|\gamma|=d$, $2d>|\theta|\geq d$ and $2d>|\theta_1|\geq d$ we define

$$g_t(\theta\gamma,\theta_1)\stackrel{\text{def}}{=}\mathsf{P}\big\{\xi(t)=\theta\gamma,\ \ |\xi(s)|\geq 2d\ \text{ for all } s \text{ with } t\geq s\geq 1\mid \xi(0)=\theta_1\big\}.$$

The probability $g_t(\theta\gamma,\theta_1)$ is a taboo probability and the following series is finite

$$g(\theta\gamma,\theta_1)\stackrel{\text{def}}{=}\sum_t g_t(\theta\gamma,\theta_1). \tag{2.8}$$

Theorem 2.2. *For all* $\alpha=\theta\gamma_1\dots\gamma_n$, *with* $2d>|\theta|\geq d$ *and* $|\gamma_1|=\dots=|\gamma_n|=d$,

$$p_\mu(\alpha)=p_\mu(\theta\gamma_1\dots\gamma_n)=\sum_{2d>|\theta_i|\geq d}g(\theta\gamma_1,\theta_1)g(\theta_1\gamma_2,\theta_2)\dots g(\theta_{n-1}\gamma_n,\theta_n)p_\mu(\theta_n). \tag{2.9}$$

This theorem will be proved in Section 3. Formula (2.9) yields a set of equations for $p_\mu(\theta)$, $2d > |\theta| \geq d$. Define the positive matrix

$$H = \Big\{ h(\theta_1, \theta_2) = \sum_{|\gamma|=d} g(\theta_1\gamma, \theta_2) \Big\}_{\substack{\theta_1,\theta_2 : 2d > |\theta_1| \geq d, \\ 2d > |\theta_2| \geq d}} \tag{2.10}$$

Corollary 2.1. *The vector* $p_\mu = \{p_\mu(\theta), \theta : 2d > |\theta| \geq d\}$ *is a right eigenvector of* H *with eigenvalue* 1*:*

$$p_\mu = Hp_\mu.$$

To see this, we use (2.9) for $n = 1$ to write

$$p_\mu(\theta) = \sum_{|\gamma|=d} p_\mu(\theta\gamma) = \sum_{\substack{|\gamma|=d, \\ 2d>|\theta_1|\geq d}} g(\theta\gamma, \theta_1)\, p_\mu(\theta_1) = \sum_{2d>|\theta_1|\geq d} h(\theta, \theta_1)\, p_\mu(\theta_1).$$

Remark 2.2. In the transient case the spectral radius of H is equal to 1.

Next we formulate a new stabilisation law for transient strings. Denote by $\xi_{[m,l]}(t)$, where $m < l$, the subsequence $\xi_m(t) \dots \xi_l(t)$ of the sequence $\xi(t) = \xi_{a_t}(t) \dots \xi_{-1}(t)$, where $a_t = - \mid \xi(t) \mid$.

Theorem 2.3. *For all* $\alpha \in \mathcal{A}$, $\alpha \neq \emptyset$, *and for all initial states* $\beta \in \mathcal{A}$, *the limit*

$$\lim_{N\to\infty} \lim_{t\to\infty} \mathsf{P}\{\xi_{[-N-|\alpha|,-N]}(t) = \alpha \mid \xi(0) = \beta\} = p_\nu(\alpha)$$

exists, where $p_\nu(\alpha)$, $\alpha \in \mathcal{A}$, *are the correlation functions of some measure* $\nu \in \mathcal{P}_\infty$ *different from* μ. *These correlation functions* $p_\nu(\alpha)$ *can be determined in the following way. For all* $\alpha \in \mathcal{A}$ *such that* $\alpha = \gamma_1 \dots \gamma_n$ *and* $|\gamma_1| = \dots = |\gamma_n| = d$,

$$p_\nu(\alpha) = p_\nu(\gamma_1 \dots \gamma_n) = \sum_{2d>|\theta_i|\geq d} \tilde{p}(\theta_1) g(\theta_1\gamma_1, \theta_2) \dots g(\theta_n\gamma_n, \theta_{n+1}) f(\theta_{n+1}), \tag{2.11}$$

where $\tilde{p} = \{\tilde{p}(\theta), \theta : 2d > |\theta| \geq d\}$ *and* $f = \{f(\theta), \theta : 2d > |\theta| \geq d\}$ *are left and right eigenvectors of* H

$$\tilde{p} = \tilde{p}H, \qquad f = Hf,$$

such that

$$\sum_{|\theta|=d} \tilde{p}(\theta) f(\theta) = 1.$$

This theorem will be proved in Section 4.

Remark 2.3. It is evident that from expression (2.11) one can derive the correlation functions $p_\nu(\alpha)$ for strings of arbitrary length.

Corollary 2.2. *The measure* ν *is a translation invariant measure with the exponentially strong mixing property.*

Thus we have defined two measures $\mu = \mu(Q_l), \nu = \nu(Q_l)$, where Q_l is the matrix $(q_l(\gamma, \delta))$, as well as velocity $v_l(\mu)$. We say that the measure $\mu = \mu(Q_l)$ is generated by the left end of the string $\xi(t)$.

2.4.2. Stabilisation law for semi-infinite strings

The results of the previous section can be reformulated for semi-infinite strings. Consider the Markov process $\mathcal{L}_\infty$ on the set of all semi-infinite strings. Suppose that the corresponding Markov chain $\mathcal{L}_l$ on the set of all finite strings is transient. Then there exists a unique invariant measure for $\mathcal{L}_\infty$ (see Definition 2.2) and it coincides with the above measure μ defined by Theorem 2.1.

The following theorem has been proved in [3].

Theorem 2.4. *Assume Conditions* 2.1, 2.4 *and assume that the Markov chain* $\mathcal{L}_l$ *transient. Then the following statements hold.*
(i) *The Markov chain* $\mathcal{L}_\infty$ *has a unique invariant measure that coincides with the measure* μ *defined by formula* (2.9).
(ii) *For any initial distribution* φ

$$\varphi P(t) \to \mu, \;\; t \to \infty, \tag{2.12}$$

in the sense of weak convergence on $\mathcal{B}_\infty$.
(iii) *For any initial distribution* φ, *convergence of the correlation functions in* (2.12) *is exponentially quickly, i.e.*

$$|p_t(\alpha \mid \varphi) - p_\mu(\alpha)| \le C(|\alpha|)e^{-\chi t} \tag{2.13}$$

for some $\chi > 0$ *not depending on* $|\alpha|$ *and some constant* $C(|\alpha|)$ *depending only on* $|\alpha|$.

2.4.3. Stabilisation law for semi-infinite strings

Consider the Markov chain $\mathcal{L}_\infty$ corresponding to the ergodic Markov chain $\mathcal{L}_l$. Recall that $(\eta(t), a_t)$ denotes the state of $\mathcal{L}_\infty$ at time t. Denote by (ψ, δ_0) the initial distribution of $\mathcal{L}_\infty$, where $\psi \in \mathcal{P}_\infty$. Note that in the ergodic case $a_t \to \infty$ a.s. for any initial distribution.

For any measure $\psi \in \mathcal{P}_\infty$ and for all natural numbers n we define the shifted measure $\Theta_n \psi$ with the following correlation functions

$$p_{\Theta_n \psi}(\gamma) = \sum_{\theta:\, |\theta| = n} p_\psi(\theta\gamma).$$

Define an infinite, strictly increasing sequence of random moments

$$0 = \sigma^{(0)} < \sigma^{(1)} < \sigma^{(2)} < \cdots < \sigma^{(n)} < \cdots$$

such that

$$\sigma^{(n)} = \min(t > \sigma^{(n-1)} : \; a_t > a_{\sigma^{(n-1)}}).$$

We will call the moments $\sigma^{(n)}$ renewal times. If $\eta(\sigma^{(n-1)}) = \alpha\rho$, for some $|\alpha| = d$, then for all n the distribution of $\sigma^{(n)} - \sigma^{(n-1)}$ depends only on α. By definition, put

$$e_\alpha = \mathsf{E}(\sigma^{(n)} - \sigma^{(n-1)} \mid \eta(\sigma^{(n-1)}) = \alpha\rho).$$

By ergodicity the above expectation is finite.

Consider the imbedded Markov chain $\mathcal{L}'_\infty$ defined by

$$\tilde{\eta}(n) = \eta(\sigma^{(n)}).$$

We will see that the following result holds.

Lemma 2.1. *The Markov process $\mathcal{L}'_\infty$ has a continuum of extremal invariant measures, which can be obtained as follows. Let the initial distribution ψ satisfy the condition*

$$\Theta_n \psi \to \varphi, \ \ n \to \infty,$$

where φ is an ergodic measure on $\mathcal{B}_\infty$. Then

$$\frac{1}{N} \sum_{n=1}^{N} \psi \tilde{P}(n) \to \pi_\varphi, \tag{2.14}$$

in the sense of weak convergence on $\mathcal{B}_\infty$, where $\tilde{P}(n)$ is the Markov semigroup on $\mathcal{P}_\infty$ corresponding to $\mathcal{L}'_\infty$.

Remark 2.4. In case $d = 1$, the distribution of $\tilde{\eta}(n)$ coincides with φ for all n and thus $\pi_\varphi \equiv \varphi$. In case $d > 1$, this distribution will be obtained below in the proof of this lemma.

For all α, β and ρ, with $|\alpha| = d, |\beta| \geq d$ and $\rho \in \mathcal{B}_\infty$, define

$$w_t(\beta, \alpha) = \mathsf{P}\{\eta(t) = \beta\rho, a_s \leq -d, \, 1 \leq s \leq t \mid \eta(0) = \alpha\rho, a_0 = -d\}. \tag{2.15}$$

The probability $w_t(\beta, \alpha)$ is well-defined, since by condition 2.1 the probability on the right-hand side of (2.15) does not depend on ρ. Define also

$$w(\beta, \alpha) = \sum_{t=0}^{\infty} w_t(\beta, \alpha) < \infty. \tag{2.16}$$

Let us write

$$\bar{e}_\varphi = \sum_{\alpha\colon |\alpha| = d} \pi_\varphi(\alpha) e_\alpha. \tag{2.17}$$

Then the following result on invariant measures of the Markov process $\mathcal{L}_\infty$ holds.

Theorem 2.5. *Assume that Conditions* 2.1, 2.4 *hold and that the Markov chain $\mathcal{L}_l$ is ergodic. Then the following assertions hold.*
(i) *For any initial distribution ψ such that*

$$\Theta_n \psi \to \varphi, \quad n \to \infty, \tag{2.18}$$

with φ *an ergodic measure on* $\mathcal{B}_\infty$*, we have*

$$\frac{1}{T}\sum_{t=1}^{T}\psi P(t) \to \kappa_\varphi \tag{2.19}$$

as $T \to \infty$*, in the sense of weak convergence on* $\mathcal{B}_\infty$*.*

The Markov process $\mathcal{L}_\infty$ *has a continuum of extremal invariant measures. The correlation functions of* κ_φ *are given by the following formula: for* $|\gamma| \geq d$

$$p_{\kappa_\varphi}(\gamma) = \frac{1}{\bar{e}_\varphi}\Big(\sum_{\alpha:\,|\alpha|=d}\pi_\varphi(\alpha)\sum_{\beta\in\mathcal{A}}w(\gamma\beta,\alpha) + \sum_{\substack{\gamma',\gamma'':\gamma=\gamma'\gamma''\\ |\gamma'|\geq d,\,\gamma''\neq\emptyset}}\ \sum_{\alpha:\,|\alpha|=d}\pi_\varphi(\alpha\gamma'')w(\gamma',\alpha)\Big), \tag{2.20}$$

where π_φ *is defined by formula* (2.14)*.*

(ii) *For any initial distribution* ψ *satisfying condition* (2.18)

$$\frac{\mathsf{E}a_T}{T} \to v_l(\kappa_\varphi) > 0, \quad T \to \infty,$$

where

$$v_l(\kappa_\varphi) = \sum_{\gamma:|\gamma|=d} p_{\kappa_\varphi}(\gamma)\sum_{\theta:|\theta|\leq 2d}(|\theta|-d)\,q_l(\gamma,\theta). \tag{2.21}$$

This theorem will be proved in Section 5.

2.5. Strings with independent evolution of both ends

Ergodicity and transience conditions for the Markov chain $\mathcal{L}$ can be obtained from the foregoing results. The above stated stabilisation laws play a main role in this classification. For example, if the right string is ergodic and the left string is transient then we need

- the invariant measure μ of the left end and the speed $v_l(\mu)$ at which the left string drifts off to infinity;
- the "induced tail measure" κ_ν, defined by the initial distribution ν, which the left string leaves behind itself (to the right of its left end);
- the speed $v_r(\kappa_\nu)$ of the right string in the environment (initial condition) of this "induced tail process" κ_ν.

Theorem 2.6. *The following classification holds.*

(i) *If both left and right strings are ergodic then* $\mathcal{L}$ *is ergodic.*

(ii) *If both left and right strings are transient then* $\mathcal{L}$ *is transient.*

(iii) *Assume that the left string is transient and the right string is ergodic. The parameters of the left string are* μ, $v_l(\mu)$*, the parameters of the right string are* κ_ν, $v_r(\kappa_\nu)$*. Then* $\mathcal{L}$ *is ergodic if*

$$v_l(\mu) < v_r(\kappa_\nu)$$

and transient if

$$v_l(\mu) > v_r(\kappa_\nu)\,.$$

(iv) *The case when the left string is ergodic and the right one is transient is symmetric to the previous one.*

This theorem will be proved in Section 6. The intuition behind the theorem is the following. Suppose that the left end of the string is transient and the right one is ergodic. Then by Theorem 1 the transient end will move with velocity $v_l(\mu)$ and by Theorem 3 it will generate a stationary medium with distribution ν in the middle of the string. The ergodic end will move in this stationary medium and by Theorem 6 its velocity will be equal to $v_r(\kappa_\nu)$. So we only have to compare the velocities of the left and right ends. If $v_l(\mu) < v_r(\kappa_\nu)$, the right end will overtake the left one and hence the string is ergodic. If $v_l(\mu) > v_r(\kappa_\nu)$, the left end will escape from the right one and so the string is transient.

3. Proof of Theorem 2.2

For $\mathsf{P}\{\xi(t) = \alpha \mid \alpha_0\} = \mathsf{P}\{\xi(t) = \alpha \mid \xi(0) = \alpha_0\}$ we have the following result.

Lemma 3.1. *For all θ, γ, ρ and α_0 with $|\gamma| = d$, $2d > |\theta| \geq d$ and $|\rho| > |\alpha_0|$,*

$$\mathsf{P}\{\xi(t) = \theta\gamma\rho \mid \alpha_0\} = \sum_{\substack{t_0+t_1=t \\ 2d>|\theta_1|\geq d}} g_{t_0}(\theta\gamma, \theta_1)\mathsf{P}\{\xi(t_1) = \theta_1\rho \mid \alpha_0\}. \tag{3.1}$$

Proof. We denote by Γ a trajectory of the Markov chain $\mathcal{L}_l$,

$$\Gamma = \alpha^{(0)}, \dots, \alpha^{(t)}$$

and by $P(\Gamma)$ probability of this trajectory

$$P(\Gamma) = p_{\alpha_0\alpha_1} \dots p_{\alpha_{t-1}\alpha_t},$$

where $p_{\cdot\,\cdot}$ are the transition probabilities of $\mathcal{L}_l$. In terms of trajectories we have

$$\mathsf{P}\{\xi(t) = \theta\gamma\rho \mid \alpha_0\} = {\sum}' P(\Gamma),$$

where the summation is over all trajectories $\Gamma = \alpha^{(0)}, \dots, \alpha^{(t)}$ with

$$\alpha^{(0)} = \alpha_0, \quad \alpha^{(t)} = \theta\gamma\rho.$$

With each trajectory Γ we can uniquely associate a moment $\sigma(\Gamma)$ such that $\alpha^{(\sigma(\Gamma))} = \theta_1\rho$ for some θ_1 with $2d > |\theta_1| \geq d$ and $|\alpha^{(s)}| \geq 2d + |\rho|$, for all $s > \sigma(\Gamma)$.

So the trajectory Γ can be decomposed into two sub-trajectories $\Gamma = \Gamma_0\Gamma_1$, where $\Gamma_0 = \alpha^{(\sigma(\Gamma)+1)}, \dots, \alpha^{(t)}$ and $\Gamma_1 = \alpha^{(0)}, \dots, \alpha^{(\sigma(\Gamma))}$.

This implies that

$$\sum{}' P(\Gamma) = \sum_{\Gamma=\Gamma_0\Gamma_1} P(\Gamma_0)P(\Gamma_1) = \sum_{\theta_1}\left[\sum{}^{(0)} P(\Gamma_0)\sum{}^{(1)} P(\Gamma_1)\right],$$

where $\sum^{(0)}$ is the sum over all $\Gamma_0 = \alpha^{(0)}, \ldots, \alpha^{(t_0)}$ such that

$$\alpha^{(0)} = \theta_1\rho, \quad \alpha^{(t_0)} = \theta\gamma\rho,$$

and

$$|\alpha^{(s)}| \geq 2d + |\rho|, \text{ for all } s > 0,$$

and $\sum^{(1)}$ is the sum over all $\Gamma_1 = \alpha^{(0)} \ldots \alpha^{(t_1)}$ such that

$$\alpha^{(0)} = \alpha_0, \quad \alpha^{(t_1)} = \theta_1\rho.$$

It is clear that

$$\sum{}^{(1)} P(\Gamma_1) = \mathsf{P}\{\xi(t_1) = \theta_1\rho \mid \alpha_0\}.$$

As the length of strings in trajectory Γ_0 is not less than $2d + |\rho|$ then the probability $P(\Gamma_0)$ does not depend on ρ. Hence the sum $\sum^{(0)}$ does not depend on ρ and

$$\sum{}^{(0)} P(\Gamma_0) = g_{t_0}(\theta\gamma, \theta_1).$$

□

This lemma has the following consequence.

Corollary 3.1. *Let $|\rho| > |\alpha_0|$,$2d > |\theta| \geq d$ and $|\gamma_1| = \ldots = |\gamma_n| = d$. Then*

$$\mathsf{P}\{\xi(t) = \theta\gamma_1\ldots\gamma_n\rho \mid \alpha_0\} \tag{3.2}$$
$$= \sum_{\substack{t_0+\cdots+t_n=t,\\ 2d>|\theta_i|\geq d}} g_{t_0}(\theta\gamma_1, \theta_1)g_{t_1}(\theta_1\gamma_2, \theta_2)\ldots g_{t_{n-1}}(\theta_{n-1}\gamma_n, \theta_n)\mathsf{P}\{\xi(t_n) = \theta_n\rho \,|\, \alpha_0\}.$$

Using this corollary we obtain the following relation

$$\begin{aligned}
&p_t(\theta\gamma_1\ldots\gamma_n \mid \alpha_0)\\
&\quad= \sum_{\rho:|\rho|>|\alpha_0|} \mathsf{P}\{\xi(t) = \theta\gamma_1\ldots\gamma_n\rho \,|\, \alpha_0\} + \sum_{\rho:|\rho|\leq|\alpha_0|} \mathsf{P}\{\xi(t) = \theta\gamma_1\ldots\gamma_n\rho \,|\, \alpha_0\}\\
&\quad= \sum_{\rho:|\rho|>|\alpha_0|} \mathsf{P}\{\xi(t) = \theta\gamma_1\ldots\gamma_n\rho \,|\, \alpha_0\} + O(\mathsf{P}\{|\xi(t)| \leq |\alpha_0| + d(n+2) \,|\, \alpha_0\})\\
&\quad= \sum_{\substack{t_0+\cdots+t_n=t,\\ 2d>|\theta_i|\geq d}} g_{t_0}(\theta\gamma_1, \theta_1)g_{t_1}(\theta_1\gamma_2, \theta_2)\ldots g_{t_{n-1}}(\theta_{n-1}\gamma_n, \theta_n) \times\\
&\qquad\quad \times \sum_{\rho:|\rho|>|\alpha_0|} \mathsf{P}\{\xi(t_n) = \theta_n\rho \,|\, \alpha_0\} + O(\mathsf{P}\{|\xi(t)| < |\alpha_0| + d(n+2)|\alpha_0\}).
\end{aligned} \tag{3.3}$$

It is clear that

$$\sum_{\rho:|\rho|>|\alpha_0|} \mathsf{P}\{\xi(t_n) = \theta_n\rho \mid \alpha_0\} = p_{t_n}(\theta_n \mid \alpha_0) + O(\mathsf{P}\{|\xi(t)| \le |\alpha_0| \mid \alpha_0\}).$$

By virtue of Theorem 2.1, the following limit exists for any α

$$p_\mu(\alpha) = \lim_{t\to\infty} p_t(\alpha \mid \alpha_0)$$

and this limit does not depend on α_0. Taking the limit in (3.3) yields (2.9). □

4. Proof of Theorem 2.3

We can use (3.2) to determine the probability

$$\hat{p}_t(\rho \mid \alpha_0) \stackrel{\text{def}}{=} \mathsf{P}\{\xi(t) = \beta\rho, \text{for some } \beta \,|\, \xi_0 = \alpha_0\}.$$

Let $|\gamma| = d$ and let $|\rho| > |\alpha_0|$. Then

$$\begin{aligned} \hat{p}_t(\gamma\rho \,|\, \alpha_0) &= \sum_{\substack{2d>|\theta|\ge d,\\ n\ge 0,|\gamma_i|=d}} p_t(\theta\gamma_1\dots\gamma_n\gamma\rho \mid \alpha_0) + \sum_{|\theta|<d} p_t(\theta\gamma\rho \,|\, \alpha_0) \\ &= \sum_{\substack{2d>|\theta|\ge d,\\ n\ge 0,|\gamma_i|=d}} p_t(\theta\gamma_1\dots\gamma_n\gamma\rho \,|\, \alpha_0) + O(\mathsf{P}\{|\xi(t)| < 2d + |\rho| \,|\, \xi(0) = \alpha_0\}) \\ &= \sum_{\substack{|\theta|,|\theta_1|\ge d,\\ t_0+t_1+t_2=t}} \tilde{p}_{t_0}(\theta) g_{t_1}(\theta\gamma, \theta_1) \mathsf{P}\{\xi(t_2) = \theta_1\rho \,|\, \alpha_0\} \\ &\quad + O(\mathsf{P}\{|\xi(t)| < 2d + |\rho| \,|\, \xi(0) = \alpha_0\}), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \tilde{p}_t(\theta) &= \sum_{n>0} \sum_{\substack{t_0+\dots+t_n=t,\\ 2d>|\theta_i|\ge d,\\ |\gamma_i|=d}} g_{t_0}(\theta_0\gamma_1, \theta_1)\dots g_{t_{n-1}}(\theta_{n-1}\gamma_n, \theta) \\ &= \mathsf{P}\{\text{for all } l : t \ge l \ge 1, |\xi(l)| \ge 2d \mid \xi(0) = \theta\}. \end{aligned} \tag{4.2}$$

In the transient case the limit exists and is positive:

$$\tilde{p}(\theta) = \lim_{t\to\infty} \tilde{p}_t(\theta) = \mathsf{P}\{\text{for all } l \ge 1, |\xi(l)| \ge 2d \mid \xi(0) = \theta\}.$$

In this case (4.2) implies that $\tilde{p}(\theta)$ satisfies the equation

$$\tilde{p}(\theta) = \sum_{\substack{2d>|\theta_1|\ge d,\\ |\gamma|=d}} \tilde{p}(\theta_1)\, g(\theta_1\gamma, \theta). \tag{4.3}$$

In the transient case clearly the sum

$$F(\rho \mid \alpha_0) = \sum_t \mathsf{P}\{\xi(t) = \rho \,|\, \alpha_0\}$$

is finite. It follows from (3.1) that

$$F(\theta\gamma\rho \mid \alpha_0) = \sum_{2d>|\theta_1|\geq d} g(\theta\gamma, \theta_1)\, F(\theta_1\rho \mid \alpha_0). \tag{4.4}$$

Taking the limit in (4.1), we obtain

$$\begin{aligned}\hat{p}(\gamma\rho \,|\, \alpha_0) = \lim_{t\to\infty} \hat{p}(\gamma\rho \,|\, \alpha_0) &= \sum_{2d>|\theta_i|\geq d} \tilde{p}(\theta) \sum_{t_1} g_{t_1}(\theta_0\gamma, \theta_1) \sum_{t_2} \mathsf{P}\{\xi(t_2) = \theta_1\rho \,|\, \alpha_0\} \\ &= \sum_{2d>|\theta_i|\geq d} \tilde{p}(\theta_0) g(\theta_0\gamma, \theta_1) F(\theta_1\rho \,|\, \alpha_0).\end{aligned}$$

Using this fact and formula (4.4), we find for $|\rho| > |\alpha_0|$ and $|\gamma_1| = \cdots = |\gamma_n| = d$ that

$$\hat{p}(\gamma_1 \ldots \gamma_n\rho \,|\, \alpha_0) = \sum_{2d>|\theta_i|\geq d} \tilde{p}(\theta_1) g(\theta_1\gamma_1, \theta_2) \ldots g(\theta_n\gamma_n, \theta_{n+1}) F(\theta_{n+1}\rho|\alpha_0).$$

Choose $m > |\alpha_0|$. Let $N = dn + m$ and $|\gamma| = d$. Then

$$\begin{aligned}\lim_{t\to\infty} \mathsf{P}\{\xi_{[-|\gamma|-N,-N]}(t) = \gamma|\xi(0) = \alpha_0\} &= \sum_{\substack{|\gamma_i|=d,\\ |\rho|=m}} \hat{p}(\gamma\gamma_1 \ldots \gamma_n\rho \mid \alpha_0) \\ = \sum_{\substack{2d>|\theta_i|\geq d,\\ |\gamma_i|=d}} \tilde{p}(\theta_0) g(\theta_0\gamma, \theta_1) g(\theta_1\gamma_1, \theta_2) \ldots g(\theta_n\gamma_n, \theta_{n+1}) &\sum_{|\rho|=m} F(\theta_{n+1}\rho \mid \alpha_0).\end{aligned}$$

The last summation can be expressed in terms of the matrix H and so written by means

$$\begin{aligned}&\lim_{t\to\infty} \mathsf{P}\{\xi_{[-|\gamma|-N,-N]}(t) = \gamma \mid \xi(0) = \alpha_0\} \\ &\quad= \sum_{\substack{2d>|\theta_i|\geq d,\\ |\gamma_i|=d}} \tilde{p}(\theta_0)\, g(\theta_0\gamma, \theta_1)\, h^{(n)}_{\theta_1\theta_2} \sum_{|\rho|=m} F(\theta_2\rho \mid \alpha_0).\end{aligned}$$

We already saw that $\tilde{p}H = \tilde{p}$ (cf. (4.3)), i.e. the spectral radius of H is equal to 1. Hence, by Perron-Frobenius' theorem the following limit exists and does not depend on α_0

$$\lim_{N\to\infty} \lim_{t\to\infty} \mathsf{P}\{\xi_{[-|\gamma|-N,-N]}(t) = \gamma \mid \xi(0) = \alpha_0\} = \sum_{2d>|\theta_i|\geq d,} \tilde{p}(\theta_0)\, g(\theta_0\gamma, \theta_1)\, f(\theta_1),$$

where $f = \{f(\theta), \theta : 2d > |\theta| \geq d\}$ is the eigenvector of H that satisfies the normalisation condition

$$\sum_{|\theta|=d} \tilde{p}(\theta)\, f(\theta) = 1.$$

5. Proof of Theorem 2.5

Let us prove assertion (i). First we will discuss the issue of invariant measures of the imbedded Markov chain $\mathcal{L}'_\infty$. Suppose that $\eta(0)$ has distribution $\psi \in \mathcal{P}_\infty$ such, that

$$\Theta_n \psi \to \varphi, \quad n \to \infty, \tag{5.1}$$

where φ is some ergodic measure on $\mathcal{B}_\infty$. We denote by $\tilde{\mathsf{P}}\{C \mid \varphi\}$ the probability of event C related to the process $\mathcal{L}'_\infty$, if the initial state has distribution φ.

Proof of Lemma 2.1. Consider the Markov chain $\mathcal{L}'_\infty$. This Markov chain jumps in one step from state $\gamma\rho$ to state $\theta\rho$, for some γ, ρ, θ with $|\gamma| = d$, $|\theta| < d$ and $\rho \in \mathcal{B}_\infty$. The probability of the transition $\gamma\rho \to \theta\rho$ does not depend on ρ and depends only on γ and θ. Denote this probability by $\tilde{p}(\gamma, \theta)$. By Condition 2.3 all probabilities $\tilde{p}(\gamma, \theta)$ are non-zero. Consider the random moments for the process $\mathcal{L}'_\infty$

$$0 = \tilde{\sigma}^{(0)} < \tilde{\sigma}^{(1)} < \tilde{\sigma}^{(2)} < \cdots < \tilde{\sigma}^{(k)} < \cdots$$

at which transitions of the type $\gamma\rho \to \emptyset\rho$ occur. The transition $\gamma\rho \to \emptyset\rho$ occurs with a probability not less than $\varepsilon = \min_{|\gamma|=d} \tilde{p}(\gamma, \emptyset) > 0$ and the conditional expectation

$$\tilde{e}_\rho = \mathsf{E}(\tilde{\sigma}^{(k+1)} - \tilde{\sigma}^{(k)} \mid \tilde{\eta}(\tilde{\sigma}^{(k)}) = \rho) < \varepsilon^{-1}$$

uniformly over $\rho \in \mathcal{B}_\infty$. By (5.1) the distribution of $\tilde{\eta}(\tilde{\sigma}^{(k)})$ tends to φ as $k \to \infty$, in the sense of weak convergence.

Let

$$K_N = \sum_{k=0}^{\infty} \mathbf{1}_{\{\tilde{\sigma}^{(k)} \le N\}}$$

be the number of moments $\tilde{\sigma}^{(k)}$ during the segment $[0, N]$ and let

$$K_N(\gamma) = \sum_{k=0}^{\infty} \mathbf{1}_{\{\tilde{\sigma}^{(k)} \le N, \tilde{\eta}(\tilde{\sigma}^{(k)}) \in A(\gamma)\}}.$$

We will show that with probability 1

$$\frac{K_N(\gamma)}{N} \to \frac{\varphi(\gamma)}{\bar{s}}, \ \ N \to \infty, \tag{5.2}$$

where

$$\bar{s} = \int\limits_{\mathcal{B}_\infty} \varphi(d\rho)\tilde{e}_\rho.$$

Let us write

$$\frac{K_N(\gamma)}{N} = \frac{K_N(\gamma)}{K_N}\frac{K_N}{N}.$$

First we will prove that with probability 1

$$\frac{K_N}{N} \to \frac{1}{\bar{s}}, \quad N \to \infty.$$

Put

$$\tilde{\tau}^{(k)} = \tilde{\sigma}^{(k)} - \tilde{\sigma}^{(k-1)}, \quad k \geq 1.$$

Then we have

$$\frac{N}{K_N} = \frac{\tilde{\tau}^{(1)} + \tilde{\tau}^{(2)} + \cdots + \tilde{\tau}^{(K_N)} + N - \tilde{\sigma}^{(K_N)}}{K_N}.$$

Since $K_N \to \infty$ almost surely, it follows from the ergodic theorem that

$$\frac{\tilde{\tau}^{(1)} + \tilde{\tau}^{(2)} + \cdots + \tilde{\tau}^{(K_N)}}{K_N} \to \bar{s}, \text{ a.s.}$$

because we can consider $\tilde{\tau}^{(k)}$ as a function of some ergodic process with distribution φ. Hence,

$$\frac{K_N}{N} \to \frac{1}{\bar{s}}, \quad N \to \infty,$$

since $N - \tilde{\sigma}^{(K_N)} < \tilde{\tau}^{(K_N+1)}$ and so

$$\frac{N - \tilde{\sigma}^{(K_N)}}{K_N} \to 0.$$

Furthermore, the distribution of $\tilde{\eta}(\tilde{\sigma}^{(k)})$ tends to φ as $k \to \infty$, where φ is ergodic by assumption. By virtue of the ergodic theorem we have

$$\frac{K_N(\gamma)}{K_N} \to \varphi(\gamma), \quad N \to \infty.$$

Write

$$\tilde{h}_N(\gamma \mid \psi) = \mathsf{E}K_N(\gamma).$$

It follows from (5.2) that

$$\frac{\tilde{h}_N(\gamma \mid \psi)}{N} = \frac{\mathsf{E}K_N(\gamma)}{N} \to \frac{\varphi(\gamma)}{\bar{s}}, \quad N \to \infty, \tag{5.3}$$

since $K_N(\gamma)/N \leq 1$.

We can now prove the assertion of the lemma. To this end we introduce the following notation:

$$\tilde{u}_l(d\rho \mid \psi) = \sum_{k=0}^{\infty} \mathsf{P}\{\tilde{\sigma}^{(k)} = l, \, \tilde{\eta}(\tilde{\sigma}^{(k)}) \in d\rho \mid \psi\}.$$

By definition, put for all $\rho \in \mathcal{B}_\infty$ and $\gamma \in \mathcal{A}$

$$\tilde{w}_l(\gamma,\rho) = \mathsf{P}\{\tilde{\eta}(l) \in A(\gamma),\ \tilde{\tau}^{(1)} > l \mid \tilde{\eta}(0) = \rho\}$$

and

$$\tilde{w}(\gamma,\rho) = \sum_{l=0}^{\infty} \tilde{w}_l(\gamma,\rho). \tag{5.4}$$

Then we get

$$\tilde{p}_n(\gamma \mid \psi) = \sum_{l=0}^{n} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\tilde{w}_{n-l}(\gamma,\rho).$$

Consequently

$$\frac{1}{N}\sum_{n=0}^{N} \tilde{p}_n(\gamma \mid \psi) = \frac{1}{N}\sum_{n=0}^{N}\sum_{l=0}^{n} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\tilde{w}_{n-l}(\gamma,\rho).$$

Changing the order of the summation in the last formula we have

$$\frac{1}{N}\sum_{n=0}^{N} \tilde{p}_n(\gamma \mid \psi) = \frac{1}{N}\sum_{l=0}^{N} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\sum_{n=l}^{N}\tilde{w}_{n-l}(\gamma,\rho).$$

By the change of variables $k = n - l$ we obtain

$$\frac{1}{N}\sum_{n=0}^{N} \tilde{p}_n(\gamma \mid \psi) = \frac{1}{N}\sum_{l=0}^{N} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\sum_{k=0}^{N-l}\tilde{w}_k(\gamma,\rho)$$

$$= \frac{1}{N}\sum_{l=0}^{N} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\sum_{k=0}^{N}\tilde{w}_k(\gamma,\rho) - \frac{1}{N}\sum_{l=0}^{N} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\sum_{k=N-l}^{N}\tilde{w}_k(\gamma,\rho). \tag{5.5}$$

The first summation in the right-hand side of the last formula satisfies

$$\frac{1}{N}\sum_{l=0}^{N} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\sum_{k=0}^{N}\tilde{w}_k(\gamma,\rho) = \frac{1}{N}\int_{\mathcal{B}_\infty} \tilde{h}_N(d\rho \mid \psi)\sum_{k=0}^{N}\tilde{w}_k(\gamma,\rho)$$

and by formulas (5.3) and (5.4) we have

$$\frac{1}{N}\int_{\mathcal{B}_\infty} \tilde{h}_N(d\rho \mid \psi)\sum_{k=0}^{N}\tilde{w}_k(\gamma,\rho) \to \frac{1}{\bar{s}}\int_{\mathcal{B}_\infty} \varphi(d\rho)\tilde{w}(\gamma,\rho).$$

Next we show for the second summation in (5.5) that

$$\frac{1}{N}\sum_{l=0}^{N} \int_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\sum_{k=N-l}^{N}\tilde{w}_k(\gamma,\rho) \to 0,\ N \to \infty.$$

To this end define

$$\tilde{W}_N(\rho) = \sum_{k=0}^{N} \sum_{\gamma \in \mathcal{A}} \tilde{w}_k(\gamma, \rho).$$

Using the obvious estimate

$$\tilde{w}_k(\gamma, \rho) \le \sum_{\gamma \in \mathcal{A}} \tilde{w}_k(\gamma, \rho)$$

we find that

$$\frac{1}{N} \sum_{l=0}^{N} \int\limits_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi) \sum_{k=N-l}^{N} \tilde{w}_k(\gamma, \rho) \le \frac{1}{N} \sum_{l=0}^{N} \int\limits_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)(\tilde{W}_N(\rho) - \tilde{W}_{N-l}(\rho))$$

$$= \frac{1}{N} \int\limits_{\mathcal{B}_\infty} \tilde{h}_N(d\rho \mid \psi)\tilde{W}_N(\rho) - \frac{1}{N} \sum_{l=0}^{N} \int\limits_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\tilde{W}_{N-l}(\rho).$$

By virtue of (5.3)

$$\frac{1}{N} \int\limits_{\mathcal{B}_\infty} \tilde{h}_N(d\rho \mid \psi)\tilde{W}_N(\rho) \to 1.$$

Choose $\varepsilon > 0$ arbitrarily small. Then

$$\frac{1}{N} \sum_{l=0}^{N} \int\limits_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\tilde{W}_{N-l}(\rho) \ge \frac{1}{N} \sum_{l=0}^{[N(1-\varepsilon)]} \int\limits_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi)\tilde{W}_{N-l}(\rho)$$

$$\ge \frac{1}{N} \int\limits_{\mathcal{B}_\infty} \tilde{h}_{[N(1-\varepsilon)]}(d\rho \mid \psi)\tilde{W}_{[N\varepsilon]}(\rho) \to -1 + \varepsilon.$$

Hence

$$\limsup \frac{1}{N} \sum_{l=0}^{N} \int\limits_{\mathcal{B}_\infty} \tilde{u}_l(d\rho \mid \psi) \sum_{k=N-l}^{N} \tilde{w}_k(\gamma, \rho) \le \varepsilon$$

uniformly in γ. This completes the proof of the lemma. □

Remark 5.1. The correlation functions of π_φ can be written as follows:

$$p_{\pi_\varphi}(\gamma) = \frac{1}{\bar{s}} \int\limits_{\mathcal{B}} \varphi(d\rho)\tilde{w}(\gamma, \rho), \tag{5.6}$$

where

$$\bar{s} = \int\limits_{\mathcal{B}} \varphi(d\rho)\mathsf{E}\tilde{\tau}_\rho.$$

We will say that a renewal occurs at time s if $\sigma^{(n)} = s$ for some n. Let us define the following function:

$$u_s(\gamma \mid \psi) = \sum_{n=0}^{\infty} \mathsf{P}\{\sigma^{(n)} = s, \eta(s) \in A(\gamma) \mid \psi\}, \tag{5.7}$$

where $A(\gamma) = \{\beta : \beta = \gamma\rho\}$ is the set of strings with left end equal to γ. It is the probability that a renewal occurs at time s and that simultaneously the process $\mathcal{L}_\infty$ hits the set $A(\gamma)$.

Let us also introduce the following function:

$$h_t(\gamma \mid \psi) = \sum_{s=0}^{t} u_s(\gamma \mid \psi). \tag{5.8}$$

This represents the mean number of renewals before time t starting with initial distribution ψ.

In the sequel we need the following lemma.

Lemma 5.1. *If the initial state of the Markov chain $\mathcal{L}_\infty$ has distribution ψ satisfying the condition*

$$\Theta_n \psi \to \varphi, \quad n \to \infty,$$

in the sense of weak convergence, then

$$\lim_{t\to\infty} \frac{h_t(\gamma \mid \psi)}{t} = \frac{p_{\pi_\varphi}(\gamma)}{\bar{e}}, \tag{5.9}$$

where

$$\bar{e} = \sum_{a\colon |\alpha|=d} \pi_\varphi(a) e_a, \quad e_a = \mathsf{E}\tau_a.$$

Proof. The proof of the lemma is the same as the proof of (5.2) and (5.3). Details are omitted. □

Next we can prove

$$\frac{1}{T}\sum_{t=0}^{T} \mathsf{P}\{\eta(t) \in A(\gamma) \mid \psi\} \to p_{\kappa_\varphi}(\gamma),$$

where $p_{\kappa_\varphi}(\gamma)$ is defined in (2.20). The correlation functions at time t are given by the formula

$$p_t(\gamma \mid \psi) = \mathsf{P}\{\eta(t) \in A(\gamma) \mid \psi\} = \sum_{s=0}^{t} \sum_{a\colon |\alpha|=d} u_s(a \mid \psi) \sum_{\rho} w_{t-s}(\gamma\rho, \alpha)$$

$$+ \sum_{\gamma',\gamma''\colon \gamma=\gamma'\gamma''} u_s(a\gamma'' \mid \psi) w_{t-s}(\gamma', \alpha), \tag{5.10}$$

where $w_{t-s}(\beta,\alpha)$ is defined by formula (2.15). Using formula (5.10) we find

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T} p_t(\gamma \mid \psi) &= \frac{1}{T}\sum_{t=0}^{T} \mathsf{P}\{\eta(t)\in A(\gamma)\mid\psi\}\\ &= \frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{t=0}^{T}\sum_{s=0}^{t}\Big(u_s(a\mid\psi)\sum_{\rho} w_{t-s}(\gamma\rho,\alpha)\\ &\quad+\sum_{\gamma',\gamma'':\gamma=\gamma'\gamma''} u_s(a\gamma''\mid\psi)w_{t-s}(\gamma',\alpha)\Big).\end{aligned}$$

Changing the order of the summation we find

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T} p_t(\gamma \mid \psi) &= \frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}\Big(u_s(a\mid\psi)\sum_{\rho}\sum_{t=s}^{T} w_{t-s}(\gamma\rho,\alpha)\\ &\quad+\sum_{\gamma',\gamma'':\gamma=\gamma'\gamma''} u_s(a\gamma''\mid\psi)\sum_{t=s}^{T} w_{t-s}(\gamma',\alpha)\Big).\end{aligned}$$

Use the change of variables $y=t-s$ to obtain

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T} p_t(\gamma \mid \psi) &= \frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}\Big(u_s(a\mid\psi)\sum_{\rho}\sum_{y=0}^{T-s} w_{y}(\gamma\rho,\alpha)\\ &\quad+\sum_{\gamma',\gamma'':\gamma=\gamma'\gamma''} u_s(a\gamma''\mid\psi)\sum_{y=0}^{T-s} w_{y}(\gamma',\alpha)\Big)\\ &= \frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}\Big(u_s(a\mid\psi)\sum_{\rho}\sum_{y=0}^{T} w_{y}(\gamma\rho,\alpha)\\ &\quad+\sum_{\gamma',\gamma'':\gamma=\gamma'\gamma''} u_s(a\gamma''\mid\psi)\sum_{y=0}^{T} w_{y}(\gamma',\alpha)\Big)\\ &\quad-\frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}\Big(u_s(a\mid\psi)\sum_{\rho}\sum_{y=T-s}^{T} w_{y}(\gamma\rho,\alpha)\\ &\quad+\sum_{\gamma',\gamma'':\gamma=\gamma'\gamma''} u_s(a\gamma''\mid\psi)\sum_{y=T-s}^{T} w_{y}(\gamma',\alpha)\Big)\\ &= I_1(T,\gamma)+I_2(T,\gamma).\end{aligned}$$

For the first summand we have the following expression

$$I_1(T,\gamma) = \sum_{a:\,|\alpha|=d}\frac{h_T(a\mid\psi)}{T}\sum_{\rho}\sum_{y=0}^{T} w_y(\gamma\rho,\alpha)$$

$$+\sum_{\gamma',\gamma'':\gamma=\gamma'\gamma''}\sum_{a:\,|\alpha|=d}\frac{h_T(a\gamma''\mid\psi)}{T}\sum_{y=0}^{T}w_y(\gamma',\alpha).$$

By Lemma 5.1, $I_1(T,\gamma)$ tends to $p_{\kappa_\varphi}(\gamma)$, which has been defined in (2.20). We have to prove that $I_2(T,\gamma)$ tends to 0. To this end we introduce

$$G_a(t)=\sum_{s=0}^{t}\mathsf{P}\{\tau_a\geq s\}.$$

Using the obvious estimate

$$w_y(\gamma,\alpha)\ \leq \mathsf{P}\{\tau_a\geq y\},$$

we find

$$\begin{aligned}
|I_2(T,\gamma)| &\leq \frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}u_s(a\mid\psi)\sum_{y=T-s}^{T}\mathsf{P}\{\tau_a\geq y\}\\
&= \frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}u_s(a\mid\psi)\,(G_a(T)-G_a(T-s))\\
&= \sum_{a:\,|\alpha|=d}\frac{h_T(a\mid\psi)}{T}G_a(T)-\frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T}u_s(a\mid\psi)G_a(T-s)\\
&= I_1'(T)+I_2'(T).
\end{aligned}$$

By virtue of Lemma 5.1

$$I_1'(T)\to\sum_{a:\,|\alpha|=d}\frac{\pi_\varphi(a)}{\bar{e}}=1,\quad T\to\infty.$$

Choose $\varepsilon>0$ arbitrarily small. Then

$$\begin{aligned}
I_2'(T)\leq -\frac{1}{T}\sum_{a:\,|\alpha|=d}\sum_{s=0}^{T(1-\varepsilon)}u_s(a\mid\psi)G_a(T-s)\leq -\sum_{a:\,|\alpha|=d}\frac{h_{T(1-\varepsilon)}(a)}{T}G_a(T\varepsilon)\\
=-\frac{T(1-\varepsilon)}{T}\sum_{a:\,|\alpha|=d}\frac{h_{T(1-\varepsilon)}(a)}{T(1-\varepsilon)}G_a(T\varepsilon)\to -1+\varepsilon,\quad T\to\infty.
\end{aligned}$$

Thus we have proved that for arbitrarily small $\varepsilon>0$

$$\limsup_{T\to\infty}|I_2(T,\gamma)|\leq\varepsilon$$

uniformly in γ. This proves the first assertion of the theorem.

The second assertion easily follows from the first. □

6. Proof of Theorem 2.6

Let (ξ_t, a_t, b_t) be the state of the process $\mathcal{L}$ at time t, where ξ_t is finite string, $\xi_t \in \mathcal{A}$, a_t is the leftmost coordinate of the string, $a_t \in \mathbf{Z}$, and b_t is rightmost coordinate, $b_t \in \mathbf{Z}$. It is clear that $b_t - a_t = |\xi_t|$. Note that we use the notation ξ_t instead of $\xi(t)$; this is done with the purpose of making formulas more compact. To prove that process is transient it is sufficient to show that

$$\mathsf{P}\{a_t \leq \tilde{v}_{\mathrm{tr}}t - d,\, b_t \geq \tilde{v}_{\mathrm{erg}}t + d \,\text{for all } t > 0 \,|\, \xi_0 = \gamma_-\gamma_+, a_0 = -d, b_0 = d\} > 0, \tag{6.1}$$

where $-v_l(\mu) < \tilde{v}_{\mathrm{tr}} < \tilde{v}_{\mathrm{erg}} < v_r(\kappa_\nu)$ and $|\gamma_-| = |\gamma_+| = d$. Denote by (ζ_t, l_t) the state of the Markov chain $\mathcal{L}_l$ at time t and by (η_t, r_t) the state of the process $\mathcal{L}_{-\infty}$ at time t.

First we note that for any $T > 0$ and $\rho \in \mathcal{B}_{-\infty}$

$$\begin{aligned}
&\mathsf{P}\{a_t \leq \tilde{v}_{\mathrm{tr}}t - d, b_t \geq \tilde{v}_{\mathrm{erg}}t + d, \text{ for all } t, T \geq t > 0 \,|\, \xi_0 = \gamma_-\gamma_+, a_0 = -d, b_0 = d\} \\
&\quad = \sum_\alpha \mathsf{P}\{\zeta_T = \alpha,\, l_t \leq \tilde{v}_{\mathrm{tr}}t - d, \text{ for all } t,\, T \geq t > 0 | \zeta_0 = \gamma_-\} \\
&\qquad \times \mathsf{P}\{r_t \geq \tilde{v}_{\mathrm{erg}}t + d, \text{ for all } t,\, T \geq t > 0 \,|\, \eta_0 = \rho\alpha\gamma_+, r_0 = d\} \\
&\quad = \mathsf{P}\{l_t \leq \tilde{v}_{\mathrm{tr}}t - d, \text{ for all } t,\, T \geq t > 0 | \zeta_0 = \gamma_-\} \\
&\qquad \times \sum_\alpha \frac{\mathsf{P}\{\zeta_T = \alpha,\, l_t \leq \tilde{v}_{\mathrm{tr}}t - d, \text{ for all } t,\, T \geq t > 0 | \zeta_0 = \gamma_-\}}{\mathsf{P}\{\, l_t \leq \tilde{v}_{\mathrm{tr}}t - d, \text{ for all } t,\, T \geq t > 0 | \zeta_0 = \gamma_-\}} \\
&\qquad \times \mathsf{P}\{r_t \geq \tilde{v}_{\mathrm{erg}}t + d, \text{ for all } t,\, T \geq t > 0 \,|\, \eta_0 = \rho\alpha\gamma_+, r_0 = d\}.
\end{aligned}$$

Define the measure $\tilde{\nu}$ by the following correlation functions

$$\begin{aligned}
&P_{\tilde{\nu}}(\alpha[N, N + |\gamma|] = \gamma) \\
&\quad = \mathsf{P}\{\zeta_\infty[N, N + |\gamma|] = \gamma \,|\, l_t \leq \tilde{v}_{\mathrm{tr}}t - d, \text{ for all } \; t > 0, \zeta_0 = \gamma_-\},
\end{aligned}$$

for any $\gamma \in \mathcal{A}$ and $N < 0$, where $\zeta_\infty[N, N + |\gamma|] = \lim_{t\to\infty} \zeta_t[N, N + |\gamma|]$. Here $\zeta_t[N, N + |\gamma|]$ is the restriction of ζ_t to the coordinates $N, N + 1, \ldots, N + |\gamma|$; $\alpha[N, N + |\gamma|]$ is defined analogously.

Passing to the limit we get

$$\begin{aligned}
&\mathsf{P}\{a_t \leq \tilde{v}_{\mathrm{tr}}t - d, b_t \geq \tilde{v}_{\mathrm{erg}}t + d, \text{ for all } t > 0 \,|\, \xi_0 = \gamma_-\gamma_+, a_0 = -d, b_0 = d\} \\
&\quad = \mathsf{P}\{\, l_t \leq \tilde{v}_{\mathrm{tr}}t - d, \text{ for all } \; t > 0 \mid \zeta_0 = \gamma_-\} \\
&\qquad \times \mathsf{P}\{r_t \geq \tilde{v}_{\mathrm{erg}}t + d\,, \text{ for all } t,\, t > 0 \mid (\tilde{\nu}, \delta_d)\}.
\end{aligned}$$

If for any γ (6.2) (it will be proved later) holds,

$$\lim_{N\to-\infty} p_{\tilde{\nu}}(\alpha_{[N,N+|\gamma|]} = \gamma) = p_\nu(\gamma), \tag{6.2}$$

then the following result is valid.

Lemma 6.1. *Suppose that* (6.2) *holds, then*

$$\mathsf{P}\{r_t \geq \tilde{v}_{\text{erg}}t + d, \ \textit{for all } t, \, t > 0 \mid (\tilde{\nu}, \delta_d) \ \} > 0.$$

Proof. Define random moments of time $\tau_n, n = 1, 2, \dots$ by

$$\tau_1 = \min\{t : r_t < d\},$$

$$\tau_k = \min\{t : r_t < -d(k-2)\} - \tau_{k-1}, \ k = 2, 3 \dots$$

By virtue of

$$\begin{aligned} &\mathsf{P}\{r_t \geq \tilde{v}_{\text{erg}}t + d, \text{ for all } t, \, t > 0 | \ (\tilde{\nu}, \delta_d)\} \\ &\geq \ \mathsf{P}\Big\{\tau_1 + \cdots + \tau_n \geq n\frac{d}{\tilde{v}_{\text{erg}}} \text{ for all } n\Big\}. \end{aligned} \tag{6.3}$$

It is sufficient to show that

$$\mathsf{P}\Big\{\tau_1 + \cdots + \tau_n \geq n\frac{d}{\tilde{v}_{\text{erg}}} \text{ for all } n\Big\} > 0 \tag{6.4}$$

is true.

By condition (6.2) we have

$$\lim_{n\to\infty} \mathsf{E}\tau_n = \frac{d}{v_e} > \frac{d}{\tilde{v}_{\text{erg}}}.$$

Note, that the following lemma holds.

Lemma 6.2. *Denote by $\mathcal{F}_n$ the σ-algebra generated by $\{\tau_0, \dots, \tau_n\}$. Then*

$$\mathsf{E}(\tau_{k+s} \mid \mathcal{F}_k) - \mathsf{E}(\tau_{k+s}) = O(e^{-c_1 s})$$

for some $c_1 > 0$.

Hence Lemma 6.1 follows from Theorem 2.1.9 (see [1]). □

Proof of Lemma 6.2. Let $\alpha \in S^{[-\infty,d]}$ denote the initial state of η_t. We will prove that $\mathsf{E}(\tau_{k+s} \mid \mathcal{F}_k) - \mathsf{E}(\tau_{k+s}) = O(e^{-c_1 s})$ uniformly in α. Let us rewrite $\alpha = \rho\gamma_n \dots \gamma_1$, where $|\gamma_i| = d$. The state of the process at moment τ_k can be written in the form $\rho\gamma_n \dots \gamma_{k+1}\chi_{k+1}$,where χ_{k+1} is a random string such that $|\chi_{k+1}| < d$. We put $\chi_1 = \emptyset$. The distribution of τ_k and χ_{k+1} depends only on $\gamma_k\chi_k$. Hence,

$$\begin{aligned} &|\mathsf{E}(\tau_{k+s} \mid \mathcal{F}_k) - \mathsf{E}(\tau_{k+s})| \\ &\leq \ \sum_{\delta_{k+s}} \mathsf{E}(\tau_{k+s} \mid \chi_{k+s} = \delta_{k+s})\big|\mathsf{P}\{\chi_{k+s} = \delta_{k+s} \mid \mathcal{F}_k\} - \mathsf{P}\{\chi_{k+s} = \delta_{k+s}\}\big| \\ &= \ \sum_{\delta_{k+s}} \mathsf{E}(\tau_{k+s} \mid \chi_{k+s} = \delta_{k+s}) \end{aligned}$$

$$
\begin{aligned}
&\times\Big|\sum_{\delta_{k+1}}\mathsf{P}\{\chi_{k+s}=\delta_{k+s}|\chi_{k+1}=\delta_{k+1}\}\mathsf{P}\{\chi_{k+1}=\delta_{k+1}|\mathcal{F}_k\}-\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\}\Big|\\
\le\ &\sum_{\delta_{k+s}}\mathsf{E}(\tau_{k+s}\mid\chi_{k+s}=\delta_{k+s})\\
&\times\Big[\sum_{\delta_{k+1}}\big|\mathsf{P}\{\chi_{k+s}=\delta_{k+s}|\chi_{k+1}=\delta_{k+1}\}-\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\}\big|\mathsf{P}\{\chi_{k+1}=\delta_{k+1}|\mathcal{F}_k\}\Big]\\
\le\ &\max_{\delta_{k+s}}\mathsf{E}(\tau_{k+s}\mid\chi_{k+s}=\delta_{k+s})\\
&\times\max_{\delta_{k+s},\delta_{k+1}}\big|\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\mid\chi_{k+1}=\delta_{k+1}\}-\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\}\big|.
\end{aligned}
$$

So it is sufficient to prove that

$$\max_{\delta_{k+s},\delta_k}\big|\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\mid\chi_k=\delta_k\}-\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\}\big|=O(e^{-c_1 s}).$$

One can show the following relation

$$\mathsf{P}\{\chi_{k+1}=\delta_{k+1}\}=\sum_{\delta_k:|\delta_k|<d}\mathsf{P}\{\chi_k=\delta_k\}\,p_{\gamma_k}(\delta_k,\delta_{k+1}),\tag{6.5}$$

where $p_{\gamma_k}(\delta_k,\delta_{k+1})=\mathsf{P}\{\chi_{k+1}=\delta_{k+1}|\chi_k=\delta_k\}$. By virtue of (6.5) we have

$$\mathsf{P}\{\chi_{k+s}=\delta_{k+s}\}=\sum_{\delta_{k+i}:|\delta_{k+i}|<d}\mathsf{P}\{\chi_k=\delta_k\}p_{\gamma_k}(\delta_k,\delta_{k+1})\dots\,p_{\gamma_{k+s-1}}(\delta_{k+s-1},\delta_{k+s}).$$

Note that there exists $\varepsilon>0$, such that $p_\gamma(\delta,\tilde\delta)>\varepsilon$ for all γ, δ and $\tilde\delta$ and

$$\sum_{\tilde\delta}p_\gamma(\delta,\tilde\delta)=1.$$

We can hence interpret χ_t as the state of a non-homogeneous Markov chain. Consequently,

$$\big|\mathsf{P}\{\chi_{k+s}=\delta\mid\chi_k\}-\mathsf{P}\{\chi_{k+s}=\delta\mid\chi_k\}\big|=O(e^{-c_1 s}),$$

because all transition probabilities of this chain are uniformly bounded away 0.

Theorem 2.1 implies that

$$\mathsf{P}\{\,l_t\le\tilde v_{\mathrm{tr}}t-d,\ \text{ for all }\ t>0\mid\zeta_0=\gamma_-\}>0,$$

and so inequality (6.1) holds.

It remains to check (6.2). To this end we will use the coupling method. Introduce new random processes $\tilde\zeta_t$ such that

$$
\begin{aligned}
&\mathsf{P}\{\tilde\zeta_{t_1}=x_1,\tilde\zeta_{t_2}=x_2,\dots,\tilde\zeta_{t_n}=x_n\}\\
&\quad=\ \mathsf{P}\{\zeta_{t_1}=x_1,\zeta_{t_2}=x_2,\dots,\zeta_{t_n}=x_n|A_t\},
\end{aligned}
$$

for all $t_1, \dots, t_n \le t, x_1, \dots, x_n$, where A_t is the event $\{l_s \le \tilde{v}_{\rm tr} s - d$, for all $t \ge s > 0\}$. Obviously this process generates the measure $\tilde{\nu}$. Denote by $\tilde{l}_t$ the leftmost coordinate of the string $\tilde{\zeta}_t$.

Let us note that $|\gamma\rho| \ge \tilde{v}_{\rm tr} t + 2d$, if $\alpha = \gamma\rho$ and $|\gamma| = d$. Then

$$\begin{aligned}
\mathsf{P}\{\tilde{\zeta}_{t+1} = \theta\rho \mid \tilde{\zeta}_t = \gamma\rho\} &= \frac{\mathsf{P}\{\tilde{\zeta}_{t+1} = \theta\rho, \tilde{\zeta}_t = \gamma\rho\}}{\mathsf{P}\{\tilde{\zeta}_t = \gamma\rho\}} \\
&= \frac{\mathsf{P}\{\zeta_{t+1} = \theta\rho, \zeta_t = \gamma\rho \mid A_{t+1}\}}{\mathsf{P}\{\zeta_t = \gamma\rho \mid A_t\}} \\
&= \mathsf{P}\{\zeta_{t+1} = \theta\rho \mid \zeta_t = \gamma\rho\} \frac{\mathsf{P}\{\zeta_t = \gamma\rho \mid A_{t+1}\}}{\mathsf{P}\{\zeta_t = \gamma\rho \mid A_t\}} \\
&= q(\gamma, \theta) \frac{\mathsf{P}\{\zeta_t = \gamma\rho \mid A_{t+1}\}}{\mathsf{P}\{\zeta_t = \gamma\rho \mid A_t\}} \\
&= q(\gamma, \theta).
\end{aligned}$$

The last equality is true, since $l_t \le \tilde{v}_{\rm tr} t - 2d$ implies $|\zeta_{t+1}| \le \tilde{v}_{\rm tr} t - d$.

As a consequence, the transition probabilities of the processes $\tilde{\zeta}_t$ and ζ_t are equal if $\tilde{l}_t \le \tilde{v}_{\rm tr} t - 2d$. We can prove he following relation.

$$\begin{aligned}
&\mathsf{P}\{\text{there exists } T : \text{for all } t > T,\, \tilde{l}_t \ge \tilde{v}_{\rm tr} t - 2d \,|\, \tilde{\zeta}_T = \alpha,\, \tilde{l}_T \le \tilde{v}_{\rm tr} T - 2d\} \\
&= \mathsf{P}\{\text{there exists } T : \text{for all}\, t > T :\, l_t \ge \tilde{v}_{\rm tr} t - 2d \,|\, \zeta_T = \alpha,\, l_T \le \tilde{v}_{\rm tr} T - 2d\} \\
&> \varepsilon > 0. \qquad (6.6)
\end{aligned}$$

We construct a coupled process $(\zeta_t, \tilde{\zeta}_t)$ as follows:

0) $(\zeta_0, \tilde{\zeta}_0) = (\gamma_-, \gamma_-)$;

1) for

$$\tilde{l}_t \ge \tilde{v}_{\rm tr} t - 2d\,, l_t \ge \tilde{v}_{\rm tr} t - 2d\,, \tilde{\zeta}_t[\tilde{l}_t, \tilde{l}_t + d] = \zeta_t[l_t, l_t + d] \qquad (6.7)$$

the processes $\tilde{\zeta}_t$ and ζ_t move together according to the transition law of ζ_t, this means that

$$\mathsf{P}\{\tilde{\zeta}_{t+1} = \theta\tilde{\rho}, \zeta_{t+1} = \theta\rho \,|\, \tilde{\zeta}_t = \gamma\tilde{\rho}, \zeta_t = \gamma\rho\} = q(\gamma, \theta),$$

if $\tilde{\zeta}_t = \gamma\tilde{\rho}$ and $\zeta_t = \gamma\rho$ for $|\gamma| = d$, then for any θ with $|\theta| \le 2d$;

2) otherwise $\tilde{\zeta}_t$ and ζ_t move independently:

$$\mathsf{P}\{\tilde{\zeta}_{t+1} = \tilde{\rho}, \zeta_{t+1} = \rho \,|\, \tilde{\zeta}_t, \zeta_t\} = \mathsf{P}\{\tilde{\zeta}_{t+1} = \tilde{\rho} \,|\, \tilde{\zeta}_t\} \mathsf{P}\{\zeta_{t+1} = \rho \,|\, \zeta_t\}.$$

So the lemma is proved, if we can prove that with probability 1 there exists a moment T, such that $\zeta_t = \tilde{\zeta}_t$, for all $t > T$. Let us define coupling time τ in the following way

$$\begin{aligned}
&\tau = \min\{T : \text{for all } t > T, \\
&\quad \tilde{\zeta}_t[\tilde{l}_t, \tilde{l}_t + d] = \zeta_t[l_t, l_t + d, l_t \le \tilde{v}_{\rm tr} t - 2d, \tilde{l}_t \le \tilde{v}_{\rm tr} t - 2d\}.
\end{aligned}$$

Transience of the process ζ_t together with (6.6) imply that

$$\mathsf{P}\{\tau < \infty\} = 1.$$

This easily implies (6.2). □

6.1. Proof of Theorem 2.6 (ergodicity)

Denote by ξ_t the process with two evolving ends, and by a_t and b_t the leftmost and rightmost coordinates respectively. The initial state of the process is $x_0 = (\alpha, 0, |\alpha|)$, i.e. $a_0 = 0, \xi_0 = \alpha, b_0 = |\alpha|$. Write

$$\begin{aligned} v_{\text{tr}} &= v_l(\mu), \\ v_e &= v_r(\kappa_\nu), \\ \mathsf{E}_x\{\cdot\} &\stackrel{\text{def}}{=} \mathsf{E}\{\cdot \mid (\xi_0, a_0, b_0) = x\}. \end{aligned}$$

Define the following random variables

$$\begin{aligned} \sigma(\alpha) &= \min\{t : |\xi_t| \le K\}, \\ \tau(\alpha) &= \min\{t : b_t \le d\}, \\ \tau_{\text{tr}}(\alpha) &= \min\{t : \text{for all } s \ge t, a_s \le -K\}, \\ t(\alpha) &= \tau(\alpha) + |\xi_{\tau(\alpha)}| \frac{c}{v_e - v_{\text{tr}}}, \text{ where } 0 < c < 1. \end{aligned}$$

Let

$$A_t = \{|\xi_s| > K, \text{ for all } s \le t\}.$$

In order to prove that

$$\mathsf{E}_{x_0}\sigma(\alpha) < \infty,$$

we use Foster's criterion (see [1]). Let $\tilde{\xi}_t = \xi_{\sigma(\alpha)\wedge t}$. We prove that there exist $\varepsilon > 0, N$ and a random variable $k(\alpha)$ such that

$$\mathsf{E}(|\tilde{\xi}_{k(\alpha)+t}| - |\tilde{\xi}_t| \mid \tilde{\xi}_t = \alpha) < -\varepsilon \mathsf{E} k(\alpha)$$

for all $t > 0$ and all α with $|\alpha| > N$. To simplify our formulae we assume that $t = 0$, $a_0 = 0$ and $b_0 = |\alpha|$. Then we can rewrite the previous inequality as

$$\mathsf{E}_{x_0}(|\xi_{k(\alpha)}| - |\xi_0|) < -\varepsilon \mathsf{E} k(\alpha), \text{ where } k(\alpha) = t(\alpha)\mathbf{1}_{\{A_{t(\alpha)}\}}.$$

This inequality follows from the fact that

$$\mathsf{E}_{x_0}\mathbf{1}_{\{A_{t(\alpha)}\}}\Delta_{t(\alpha)} < -\varepsilon \mathsf{E}_{x_0} t(\alpha)\mathbf{1}_{\{A_{t(\alpha)}\}},$$

with $\Delta_{t(\alpha)} = |\xi_{t(\alpha)}| - |\xi_0.|$

The following lemma will be proved in the next section.

Lemma 6.1. *There exist $\varepsilon > 0$, $0 < c < 1$, N and K, such that for all α with $|\alpha| > N$*

i)

$$\mathsf{E}_{x_0} a_{t(\alpha)} \mathbf{1}_{\{A_{t(\alpha)}\}} = -v_{\mathrm{tr}} \Big(1 + v_{\mathrm{tr}} \frac{c}{v_e - v_{\mathrm{tr}}} \Big) \mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} + O(1);$$

ii)

$$\begin{aligned} \mathsf{E}_{x_0} b_{t(\alpha)} \mathbf{1}_{\{A_{t(\alpha)}\}} &= -v_e v_{\mathrm{tr}} \frac{c}{v_e - v_{\mathrm{tr}}} \mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} + O(1), \\ \mathsf{E}_{x_0} \mathbf{1}_{\{A_{t(\alpha)}\}} \Delta_{t(\alpha)} &= (1-c) v_{\mathrm{tr}} \mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} - |\alpha| + O(1); \end{aligned}$$

iii)

$$\mathsf{E}_{x_0} t(\alpha) \mathbf{1}_{\{A_{t(\alpha)}\}} = \mathsf{E}_{x_0} \tau(\alpha) + v_{\mathrm{tr}} \frac{c}{v_e - v_{\mathrm{tr}}} \mathsf{E}_{x_0} \tau(\alpha) + O(1).$$

So we need to prove

$$(1-c) v_{\mathrm{tr}} \mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} - |\alpha| + O(1) < -\varepsilon \mathsf{E}_{x_0} t(\alpha) \mathbf{1}_{\{A_{t(\alpha)}\}},$$

or

$$\Big((1-c) v_{\mathrm{tr}} + \varepsilon (1 + v_{\mathrm{tr}} \frac{c}{v_e - v_{\mathrm{tr}}}) \Big) \mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} + O(1) < |\alpha|.$$

For ergodicity of the process it is hence sufficient to show the following lemma.

Lemma 6.2. *There exists N such that $\mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} < N|\alpha|$, for all α.*

Proof. Define a new process $\tilde{\xi}_t$ in the following way:

1) $(\tilde{\xi}_t, \tilde{a}_t, \tilde{b}_t) = (\xi_t, a_t, b_t)$, till the moment that $|\xi_t| \geq K$;

2) if $|\xi_t| \geq K$ and $|\xi_{t+1}| < K$, then we put $\tilde{\xi}_{t+1} = \emptyset$ and $\tilde{b}_{t+1} = -K$.

It is clear that

$$\mathsf{E}_{x_0} \tau(\alpha) \mathbf{1}_{\{A_{\tau(\alpha)}\}} = \mathsf{E}_{x_0} \tilde{\tau}(\alpha),$$

where $\tilde{\tau}(\alpha) = \min\{t : \tilde{b}_t \leq d\}$. Define also random moments

$$\tau_1 = \min\{t : t > 0, \tilde{b}_t > \tilde{b}_0 + K/2 \ \text{ or } \tilde{b}_t < \tilde{b}_0 - K/2\},$$

$$\tau_{i+1} = \min\{t : t > \tau_i, \tilde{b}_t > \tilde{b}_{\tau_i} + K/2 \ \text{ or } \tilde{b}_t < \tilde{b}_{\tau_i} - K/2\}.$$

Using the Lyapounov function from [4], we can show that

$$\mathsf{P}_x\{\tilde{b}_{i+1} > \tilde{b}_{\tau_i} + K/2\} < c_1 e^{-c_2 K}, \ \text{for some } c_1, c_2 > 0 \text{ uniformly in } x$$

and

$$\mathsf{E}_x \tau_{i+1} < \infty, \ \text{uniformly in } x.$$

So there exists K such that

$$\mathsf{E}_{x_0}(\tilde{b}_{\tau_{i+1}} - \tilde{b}_{\tau_i}) < -\varepsilon, \quad \text{for some } \varepsilon > 0 \text{ uniformly in } x_0.$$

Hence, there exists N such that

$$\mathsf{E}_{x_0}\tilde{\tau}(\alpha) < N|\alpha|.$$

This completes the proof of the lemma. □

6.2. Proof of Lemma 6.1

Let us prove assertion i), the other assertions can be proved in the same way. One can write the following expression

$$\begin{aligned}\mathsf{E}_{x_0} a_{t(\alpha)} \mathbf{1}_{\{A_{t(\alpha)}\}} &= \sum_{\substack{t,\theta,\beta:\\ 2d+K>|\theta|\geq K}} \Big(\mathsf{P}_{x_0}\{\tau_{\mathrm{tr}}(\alpha) = t, \xi_t = \theta\beta, A_t\} \\ &\qquad \times \mathsf{E}_{x_0}(a_{t(\alpha)}\mathbf{1}_{\{A_{t(\alpha)}\}} \mid \tau_{\mathrm{tr}}(\alpha) = t, \xi_t = \theta\beta, A_t)\Big).\end{aligned}$$

Rewrite the conditional expectation in the last formula

$$\begin{aligned}&\mathsf{E}_{x_0}(a_{t(\alpha)}\mathbf{1}_{\{A_{t(\alpha)}\}} \mid \tau_{\mathrm{tr}}(\alpha) = t, \xi_t = \theta\beta, A_t) \\ &\qquad = \mathsf{E}_{x}\{a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}} \mid \text{for all } s \geq 0, a_s \leq -K),\end{aligned}$$

where $x = (\theta\beta, -|\theta|, |\beta|)$. We use the following notation

$$\begin{aligned}\tilde{\mathsf{E}}(\cdot) &= \mathsf{E}_x(\cdot \mid \text{for all } s \geq 0, a_s \leq -K), \\ \tilde{\mathsf{P}}\{\cdot\} &= \mathsf{P}_x\{\cdot \mid \text{for all } s \geq 0, a_s \leq -K\}.\end{aligned}$$

The proposition below immediately implies assertion i).

Proposition 6.1.

$$\tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}) = -v_{\mathrm{tr}}\Big(1 + v_{\mathrm{tr}}\frac{c}{v_e - v_{\mathrm{tr}}}\Big)\tilde{\mathsf{E}}(\tau(\theta\beta)\mathbf{1}_{\{A_{\tau(\theta\beta)}\}}) + O(1).$$

Proof of Proposition 6.1. Split the left-hand side of the above equality into two parts

$$\begin{aligned}\tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}) &= \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{a_{\tau(\theta\beta)}\in U_\varepsilon(-v_{\mathrm{tr}}\tau(\theta\beta))\}}) \\ &\quad + \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{a_{\tau(\theta\beta)}\notin U_\varepsilon(-v_{\mathrm{tr}}\tau(\theta\beta))\}}),\end{aligned}$$

where $U_\varepsilon(t) \overset{\mathrm{def}}{=} \{n \in \mathbf{Z} : |n - t| < \varepsilon t\}$, $\varepsilon > 0$. It is not difficult to prove that the second term in the right-hand side of the above equality is bounded. Let us estimate the first term. We need some notation

$$\begin{aligned} v &= (v_e + v_{\rm tr})/2, \\ L_t &= U_\varepsilon(-v_{\rm tr}\tau(\theta\beta))/2 - v(t-\tau(\theta\beta)), \\ \sigma_e &= \min\{t \geq \tau(\theta\beta) : b_t \in L_t\}, \\ \sigma_{\rm tr} &= \min\{t \geq \tau(\theta\beta) : a_t \in L_t\}, \\ T(t) &\stackrel{\rm def}{=} c\frac{(v_{\rm tr}+\varepsilon)}{v_e - v_{\rm tr}}t, \\ B_t &= \{a_t \in U_\varepsilon(-v_{\rm tr}t)\}. \end{aligned}$$

This first term reduces to

$$\begin{aligned} \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}) \;=\; & \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_{\rm tr}\wedge\sigma_e > T(\tau(\theta\beta))\}}) \\ & + \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_{\rm tr}<\sigma_e, \sigma_{\rm tr} < T(\tau(\theta\beta))\}}) \\ & + \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_e<\sigma_{\rm tr}, \sigma_e < T(\tau(\theta\beta))\}}). \end{aligned}$$

First we will show that

$$\begin{aligned} \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_{\rm tr}<\sigma_e, \sigma_{\rm tr} < T(\tau(\theta\beta))\}}) &< \infty, \\ \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_e<\sigma_{\rm tr}, \sigma_e < T(\tau(\theta\beta))\}}) &< \infty \end{aligned} \tag{6.8}$$

uniformly in $\theta\beta$.

Note that

$$a_{t(\theta\beta)} < \tilde{c}\tau(\theta\beta),$$

for some constant $\tilde{c}$.

Then for (6.8) it is sufficient to show the following bounds.

$$\tilde{\mathsf{E}}(\tau(\theta\beta)\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_{\rm tr}<\sigma_e, \sigma_{\rm tr} < T(\tau(\theta\beta))\}}) < \infty, \tag{6.9}$$

$$\tilde{\mathsf{E}}(\tau(\theta\beta)\mathbf{1}_{\{A_{\tau(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_e<\sigma_{\rm tr}, \sigma_e < T(\tau(\theta\beta))\}}) < \infty. \tag{6.10}$$

The first inequality follows directly from the existence of constants c_1, c_2, such that

$$\tilde{\mathsf{P}}\{\sigma_{\rm tr} < \sigma_e, \sigma_{\rm tr} < T(t) | B_t, \tau(\theta\beta) = t\} < c_1 e^{-c_2 t}.$$

Let us prove that inequality (6.10). By Chebyshev's inequality

$$\begin{aligned} & \tilde{\mathsf{E}}(\tau(\theta\beta)\mathbf{1}_{\{A_{\tau(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_e<\sigma_{\rm tr}, \sigma_e < T(\tau(\theta\beta))\}}) \\ & \leq \; \tilde{\mathsf{E}}(\tau(\theta\beta)\mathbf{1}_{\{\tau(\theta\beta)>|\theta\beta|^2\}}) \\ & \qquad + |\theta\beta|^2 \tilde{\mathsf{P}}\{A_{\tau(\theta\beta)}, B_{\tau(\theta\beta)}, \sigma_e < \sigma_{\rm tr}, \sigma_e < T(\tau(\theta\beta))\}. \end{aligned}$$

One can prove that

$$\tilde{\mathsf{E}}(\tau(\theta\beta)\mathbf{1}_{\{\tau(\theta\beta)>|\theta\beta|^2\}}) < \infty$$

uniformly in $\theta\beta$. A uniform bound for the second expression in the right-hand side of the above inequality is obtained from the following lemma.

Lemma 6.3. *There exist c_1, c_2 such that*

$$\tilde{\mathsf{P}}\{A_{\tau(\theta\beta)}, B_{\tau(\theta\beta)}, \sigma_e < \sigma_{\mathrm{tr}}, \sigma_e < T(t) \mid \tau(\theta\beta) = t\} < c_1 e^{-c_2 t}.$$

Clearly this lemma implies that

$$|\theta\beta|^2 \tilde{\mathsf{P}}\{A_{\tau(\theta\beta)}, B_{\tau(\theta\beta)}, \sigma_e < \sigma_{\mathrm{tr}}, \sigma_e < T(\tau(\theta\beta))\} < c_1|\theta\beta|^2 \tilde{\mathsf{E}} e^{-c_2\tau(\theta\beta)} < \infty$$

uniformly in $\theta\beta$, since $\tau(\theta\beta) \geq |\theta\beta|/d$.

Proof of Lemma 6.3. We will use the following estimate

$$\tilde{\mathsf{P}}\{A_t, B_t, \sigma_e < \sigma_{\mathrm{tr}}, \sigma_e < T(t) \mid \tau(\theta\beta) = t\} \leq \sum_{t_0 < T(t)} \tilde{\mathsf{P}}\{\sigma_e = t_0, \sigma_{\mathrm{tr}} \geq t_0 \mid \tau(\theta\beta) = t\}.$$

Note that

$$\begin{aligned} &\tilde{\mathsf{P}}\{\sigma_e = t_0, \sigma_{\mathrm{tr}} \geq t_0 \mid \tau(\theta\beta) = t\} \\ &\quad = \sum_\rho \tilde{\mathsf{P}}_\theta\{\zeta_{t_0}[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0)d, d] = \rho\} \mathsf{P}\{\sigma_e = t_0 \mid \eta_t = \rho\}, \end{aligned}$$

where the processes ζ_t, η_t have been defined in the proof of the transient case. Let us estimate the above probability:

$$\begin{aligned} &\tilde{\mathsf{P}}_\theta\big\{\zeta_{t_0}[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0)d, d] = \rho\big\} \\ &= \tilde{\mathsf{P}}_\theta\big\{\zeta_{t_0}[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho, \\ &\qquad \text{for all } t_1 \geq t_0, l_{t_1} \leq -v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - 2d\big\} \\ &\quad + \tilde{\mathsf{P}}_\theta\big\{\zeta_{t_0}[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho, \\ &\qquad \text{there exists } t_1 \geq t_0 : l_{t_1} > -v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - 2d\big\} \\ &= \tilde{\mathsf{P}}_\theta\big\{\zeta_\infty[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho\big\} \\ &\quad - \tilde{\mathsf{P}}_\theta\big\{\zeta_{t_0}[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho \\ &\qquad \text{there exists } t_1 \geq t_0 : l_{t_1} > -v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - 2d, \\ &\qquad \text{there exists } t_2 \geq t_0 : \zeta_{t_2}[-v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho, \\ &\qquad \text{for all } t_3 \geq t_2, l_{t_3} \leq -v_{\mathrm{tr}} t(1-\varepsilon)/2 - v(t-t_0) - 2d\big\} \end{aligned}$$

$$
\begin{aligned}
&+ \tilde{\mathsf{P}}_\theta\big\{\zeta_{t_0}[-v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho,\\
&\qquad \text{there exists } t_1 \geq t_0 : l_{t_1} > -v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0) - 2d\big\}\\
= \;& \tilde{\mathsf{P}}_\theta\big\{\zeta_\infty[-v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0) - d, d] = \rho\big\}\\
&+ O(\tilde{\mathsf{P}}_\theta\big\{\text{there exists } t_1 > t_0 : l_{t_1} > -v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0) - d)\big\}.
\end{aligned}
$$

So

$$
\begin{aligned}
&\tilde{\mathsf{P}}\{\sigma_e = t_0, \sigma_{\mathrm{tr}} \geq t_0 \mid \tau(\theta\beta) = t\}\\
&\quad = \sum_\rho \tilde{\mathsf{P}}_\theta\{\zeta_{t_\infty}[-v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0)d, d] = \rho\}\mathsf{P}\{\sigma_e = t_0 \mid \eta_t = \rho\}\\
&\qquad + O(\tilde{\mathsf{P}}_\theta\{\text{there exists } t_1 > t_0 : l_{t_1} > -v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0) - d)\}.
\end{aligned}
$$

The second term in the last formula is easily estimated as follows:

$$
\tilde{\mathsf{P}}_\theta\{\text{there exists } t_1 > t_0 : l_{t_1} > -v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0) - d\} < c_1 e^{-c_2 t},
$$

for some $c_1, c_2 > 0$.

We will derive a similar estimate for the first term.

$$
\begin{aligned}
&\sum_{t_0 < T(t)} \sum_\rho \tilde{\mathsf{P}}_\theta\{\zeta_{t_\infty}[-v_{\mathrm{tr}}t(1-\varepsilon)/2 - v(t-t_0)d, d] = \rho\}\mathsf{P}\{\sigma_e = t_0 \mid \eta_t = \rho\}\\
&\qquad = \sum_{t_0 < T(t)} \mathsf{P}\{\sigma_e = t_0 \mid \nu(t)\} = \mathsf{P}\{\sigma_e < T(t) \mid \nu(t)\},
\end{aligned}
$$

where $\nu(t)$ is the measure generated by the process ζ_t. Theorem 2.4 together with Corollary 6.1 below imply that we have that

$$
\mathsf{P}\{\sigma_e < T(t) \mid \nu(t)\} < c_1 e^{-c_2 t}.
$$

This proves Lemma 6.3 and thus (6.8). □

We have obtained the following result

$$
\begin{aligned}
&\tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}})\\
&\qquad = \tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_{\mathrm{tr}}\wedge\sigma_e > T(\tau(\theta\beta))\}}) + O(1).
\end{aligned}
$$

In the second expectation in the above relation the left and right ends of a string do not interact. Therefore, the evolution of the left end of the string after time $\tau(\theta\beta)$ reduces to the evolution of a string with one end only. It is not difficult to show that then

$$
\tilde{\mathsf{E}}(a_{t(\theta\beta)}\mathbf{1}_{\{A_{t(\theta\beta)}\}}\mathbf{1}_{\{B_{\tau(\theta\beta)}\}}\mathbf{1}_{\{\sigma_{\mathrm{tr}}\wedge\sigma_e > T(\tau(\theta\beta))\}}) + O(1)
$$

$$= -v_{\rm tr}(1 + v_{\rm tr}\frac{c}{v_e - v_{\rm tr}})\mathsf{E}_{x_0}\tau(\alpha)\mathbf{1}_{\{A_{\tau(\alpha)}\}} + O(1).$$

This completes the proof of Theorem 2.6. □

To prove Corollary 6.1 we need lemma:

Lemma 6.3. *Let ν be a measure on the set of semi-infinite strings $S^{[-\infty,0]}$. Let (η_t, r_t) be an ergodic string with initial distribution (ν, δ_0) such that*

$$\lim_{t\to\infty} \mathsf{E}(r_{t+1} - r_t) = -v_e.$$

Then for any $0 < c < 1$ there exist constants c_1 and c_2 such that

$$\mathsf{P}\{\textit{there exists } t : t \le c\,\frac{N}{v_e}, r_t < -N\} < c_1 e^{-c_2 N} \quad \textit{for all } N.$$

Corollary 6.1. *Suppose that the assumptions of Lemma 6.3 holds. Let $0 < v < v_e$. Then for any $0 < c < 1$ there exist constants c_1 and c_2, such that*

$$\mathsf{P}\{\textit{there exists } t : t \le c\frac{N}{v_e - v}, r_t < -N - vt\} < c_1 e^{-c_2 N} \quad \textit{for all } N > 0.$$

Proof. Letting $\tau = \min\{t : \eta_t < -N - vt\}$, we have

$$\begin{aligned}
\mathsf{P}\{\text{there exists } t : t \le c\frac{N}{v_e - v}, r_t < -N - vt\} &= \sum_{t:t\le cN/(v_e-v)} \mathsf{P}\{\tau = t\} \\
\le \sum_{t:t\le cN/(v_e-v)} &\mathsf{P}\{\text{there exists } \tilde t : \tilde t \le t, r_{\tilde t} < -N - vt\} \\
= \sum_{t:t\le cN/(v_e-v)} &\mathsf{P}\{\text{there exists } \tilde t : \tilde t \le \frac{tv_e}{N+vt}\,\frac{N+vt}{v_e}t, r_{\tilde t} < -N - vt\}.
\end{aligned}$$

For $t \le c\dfrac{N}{v_e - v}$,

$$\frac{tv_e}{N+vt} \le \frac{v_e c}{vc + v_e - v} = \frac{v_e c}{v_e c + (1-c)(v_e - v)} = \tilde c < 1.$$

Hence, by Lemma 6.3

$$\begin{aligned}
&\sum_{t:t\le cN/(v_e-v)} \mathsf{P}\{\tau = t\} \\
&\le \sum_{t:t\le cN/(v_e-v)} \mathsf{P}\{\text{there exists } \tilde t : \tilde t \le \tilde c\frac{N+vt}{v_e}t, r_{\tilde t} < -N - vt\} \\
&\le \sum_{t:t\le cN/(v_e-v)} c_1 e^{-c_2(N+vt)} < \tilde c_1 e^{-c_2 N},
\end{aligned}$$

for some constants $\tilde{c}_1, c_2 > 0$. This proves the corollary. □

Proof of Lemma 6.3. Let $N + d = dn + m, n = [N/d] + 1$. Define random moments

$$\tau_0 = \min\{t : r_t < -m\},$$

$$\tau_k = \min\{t : r_t < -dk - m\} - \tau_{k-1},\ k = 1, \ldots, n.$$

In terms of these moments we can rewrite the assertion of the lemma as follows

$$\mathsf{P}\{\text{there exists } t : t \le c\frac{N}{v_e}, r_t < -N\}$$
$$= \mathsf{P}\{\tau_0 + \cdots + \tau_n \le c\frac{N}{v_e}\} \le \mathsf{P}\{\tau_0 + \cdots + \tau_n \le cn\frac{d}{v_e}\}. \tag{6.11}$$

By assumption

$$\lim_{n\to\infty} \mathsf{E}\tau_n = \frac{d}{v_e}.$$

Denote by $\mathcal{F}_n$ σ-algebra generated by $\{\tau_0, \ldots, \tau_n\}$. Lemma 6.2 implies $\mathsf{E}(\tau_{k+s} \mid \mathcal{F}_k) - \mathsf{E}(\tau_{k+s}) = O(e^{-c_1 s})$ and so the assertion of Lemma 6.3 follows from Theorems 2.1.7 and 2.1.8 (see [1]). □

7. Examples

7.1. FIFO queues with several customer types

The queue with FIFO (First-In-First-Out) service discipline is a special case of strings with two ends. Let us consider the following discrete time example. Suppose that customers of type a, $a \in \{1, \ldots, r\}$, arrive at the left end of the queue with probability λ_a, $\sum_a \lambda_a < 1$. If there is a pair ab of custmomers at the right end of the queue, this means that the queue can be presented as a string of the form ρab. Then either this couple is served with probability $q_2(ab)$, or customer b is served with probability $q_1(ab)$, where

$$q_2(ab) + q_1(ab) < 1.$$

It follows that

$$q_r(ab, \emptyset) = q_2(ab),\ q_r(ab, a) = q_1(ab)$$

and

$$q_0(ab) \stackrel{\text{def}}{=} q_r(ab, ab) = 1 - q_2(ab) - q_1(ab).$$

The objective is to determine ergodicity conditions for this system. First of all, we have to find the measure ν on the set of semi-infinite strings $\{1, \ldots, r\}^{(-\infty,0]}$ generated by the transient end of the string. In this case it is a Bernoulli measure $\nu = \bigotimes_{-\infty}^{0} \pi$, where π is the measure on the finite set $\{1, \ldots, r\}$ defined by

$$\pi_a = \frac{\lambda_a}{\sum_a \lambda_a}.$$

To compute the "velocity of serving" (it is velocity of the ergodic end) we determine the distribution of the last two symbols by using formula (2.20). We have

$$p(ab) = \frac{1}{Z}\pi_a\pi_b w(ab, ab),$$

$$Z = \sum_{ab} \pi_a\pi_b w(ab, ab),$$

$$w(ab, ab) = \sum_{t=0}^{\infty} q_0^t(ab) = \frac{1}{q_2(ab) + q_1(ab)}.$$

It follows that

$$p(ab) = \frac{1}{Z}\pi_a\pi_b \frac{1}{q_2(ab) + q_1(ab)}.$$

Hence, the velocity of the ergodic end is equal to

$$v_e = \sum_{ab} (2q_2(ab) + q_1(ab))\, p(ab) = \frac{1}{Z}\sum_{ab} \pi_a\pi_b \frac{2q_2(ab) + q_1(ab)}{q_2(ab) + q_1(ab)}.$$

The velocity of the transient end is equal to

$$v_{\mathrm{tr}} = \sum_a \lambda_a.$$

We have obtained that the Markov chain is ergodic, if

$$\sum_a \lambda_a < \frac{1}{Z}\sum_{ab} \pi_a\pi_b \frac{2q_2(ab) + q_1(ab)}{q_2(ab) + q_1(ab)}.$$

In the special case that $q_1(ab) = q(a)(1-q(b))$, $q_2(ab) = q(a)\, q(b)$, this ergodicity condition has a simpler form:

$$\sum_a \lambda_a < \frac{\sum_a \lambda_a(1 + q(a))}{\sum_a \dfrac{\lambda_a}{q(a)}}.$$

We cannot explain this formula in terms of "standard laws" of queueing theory that state that the "load" should be less than 1.

7.2. Communication network with fixed routing

Let the network consist of two lines I and II. Calls arrive and are subsequently served at these lines. Each line has capacity 1, which means that two different calls can not use the same line at the same time. There are three types of calls in the network:

1) type a calls occupy line I;

2) type b calls occupy line II;

3) type c calls occupy lines I and II simultaneously.

All arriving calls are routed to the same queue and they are served at rates μ_a, μ_b and μ_c in FIFO order. If at the beginning of the queue type a and b calls are present, then each of them is served independently at rates μ_a and μ_b respectively. Otherwise only the call at the head of the queue is served.

It follows that the network can be interpreteted as a two-sided string. As in the previous example it is not difficult to obtain sufficient conditions for ergodicity.

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