J. Appl. Prob. **26**, 363–371 (1989)
Printed in Israel

THE STABILITY OF INFINITE-SERVER NETWORKS WITH RANDOM ROUTING

S. A. BEREZNER AND
V. A. MALYSHEV,* *Moscow State University*

Abstract

We consider networks with a very large or infinite number of nodes, linked by cable channels. The request which comes to a node is ordered to occupy a certain route of successive channels. The functioning of the system is regulated by the reserving of channels in order of the arrivals of the requests. Under some general conditions the existence of an ergodic region for such networks is proved. The practical value of the result lies in the fact that these conditions do not depend on the size of the graph.

CHANNEL-SWITCHING NETWORKS; GENERALIZED FIFO PROTOCOL; WAITING TIME PROCESS; LIMITING STATIONARY PROCESS

1. Introduction

The mathematical investigation of complex queueing or communication systems generally follows two main lines: (i) systems with a small number of servers (nodes); (ii) systems where the number of nodes is very large. Some of the latter systems are 'completely integrable' in some sense (the well-known Jackson networks provide an example). For the others the general methods of mathematical statistical physics are available (see [3]–[6]). The method used here is to consider networks with an infinite number of nodes. Stability for this case implies stability for all finite similar networks. We consider not necessarily Markovian circuit-switching networks with the FIFO protocol, which seems to be the most unstable among such protocols. We prove that stability holds for small intensity of requests uniformly in the number of nodes if the graph structure of the systems is sufficiently homogeneous.

1.1. *The graph* Γ *of the network consists of an at most countable set of vertices* C. Every vertex $c \in C$ is linked by no less than 1 and no more than ν edges with other vertices. No other restrictions on the configuration of the graph are assumed. Every edge between vertices c and c' can be designated (c, c') or (c', c) according to its direction. An ordered finite set of directed edges is called the route $\gamma(c)$ from the vertex c, if it satisfies the following conditions:

Received 11 December 1987.
* Postal address for both authors: Probability Chair, Faculty of Mathematics and Mechanics, Moscow State University, Moscow, USSR.

(i) vertex c is the beginning of the first edge;

(ii) the end of every edge (except the last edge) is the beginning of the next one.

We call the number of edges in the route $\gamma(c)$ the length of this route and denote it $|\gamma(c)|$. Γ_c denotes the set of all routes from the vertex c. Two routes $\gamma(c)$ and $\gamma(c')$ are defined to intersect (this will be written $\gamma(c) \cap \gamma(c')$), if they pass through the same vertex.

1.2. *The arrival stream U_c^0 on the vertex c will be introduced in the following way*:

(i) The arrival time (from the moment $t = 0$) of the first request on the vertex c is T_1^c. The interval between the arrivals of the nth and $(n+1)$th requests is T_n^c, $n \in N$ (the natural numbers) and $n \geqq 2$.

(ii) The service time of the nth request on the vertex c is S_n^c.

(iii) The route of the nth request on the vertex c is $\Gamma_n^c \in \Gamma_c$. The total arrival stream we denote as U^0. We denote the initial conditions W^0 as a vector of random variables $W^0[\gamma(c)]$ for $c \in C$, $\gamma(c) \in \Gamma_c$. If all the components of the vector are equal to 0 the initial condition is said to be trivial. As we see each request is uniquely determined by the triple $(t, s, \gamma(c))$, where t is its arrival time, s its service time, $\gamma(c)$ is its route with the indication of the vertex of arrival c.

1.3. The functioning of the system is organized according to the discipline, which is to a certain extent the generalization of the discipline first in, first out (FIFO). Now we shall describe it. Consider an arbitrary request $(t, s, \gamma(c))$. The request $(t', s', \gamma(c'))$ is called influencing on $(t, s, \gamma(c))$ if $t' < t$ and $\gamma(c') \cap \gamma(c)$. The service of the request $(t, s, \gamma(c))$ will begin at the moment $t_0 \geqq t$, satisfying the following conditions:

(i) all the requests influencing $(t, s, \gamma(c))$ have left the network by the time t_0.

(ii) $t_0 \geqq W^0[\gamma(c)]$.

(iii) t_0 is the smallest value satisfying (i) and (ii).

The remainder $t_0 - t$ is a waiting time $W^t[\gamma(c)]$ of the request, during which it will stay in a queue. At the moment $t_0 + s$ the request will leave the network. The functioning of the system is correctly defined if the waiting time of each request is finite. The correctness will be proved below.

2. Restrictions on the arrival stream

2.1. All the random variables $(T_n^c, S_n^c, \Gamma_n^c)$, $n \in N$, $c \in C$ are mutually independent. Variables (S_n^c), $n \in N$, $c \in C$ have the same distribution $F(s)$. Variables (T_n^c), $n \geqq 2$, $c \in C$ have distribution $\phi(t)$ and variables (T_1^c), $c \in C$ have distribution $\phi_1(t)$. Variables (Γ_n^c) $n \in N$ have the same distribution $G^c(\gamma(c)) = P\{\Gamma_n^c = \gamma(c)\}$, $\gamma(c) \in \Gamma_c$. Distributions $G^c(\gamma(c))$ are unique to each vertex $c \in C$.

2.2. We assume $F(s)$ to have exponential decrease of its 'tail', that is $\exists\, S_0, \mu > 0$ such that $\forall s \geqq S_0$

$$1 - F(s) = P\{S_n^c > s\} \leqq e^{-\mu s}. \tag{1}$$

2.3. Let $\phi(t)$ be continuous at the point $t = 0$ and $M(T_n^c)^2 < \infty$. Then we can define

(2) $$\phi_1(t) = \int_0^t (1 - \phi(u))du.$$

We now introduce some new notions. Let $n(\tau, c)$, $\tau \in R_+$ (non-negative real numbers) $c \in C$ be an integer-valued random variable such that

$$n(\tau, c) = \kappa \in N \Leftrightarrow T_1^c + \cdots + T_{n(\tau,c)}^c < \tau < T_1^c + \cdots + T_{n(\tau,c)+1}^c$$

and $n(\tau, c) = 0$ iff $T_1^c > \tau$. Then let $T_n^{\tau,c} = T_{n(\tau,c)+n}^c$, $n \in N$,

$$S_n^{\tau,c} = S_{n(\tau,c)+n}^c \quad \text{and} \quad \Gamma_n^{\tau,c} = \Gamma_{n(\tau,c)+n}^c, \qquad n \in N.$$

The corresponding arrival stream we denote by U^τ. It is well known (see [1], [2]) that for every τ, $\tau' \in R_+$ the processes U^τ and $U^{\tau'}$ are equivalent (have the same distributions), if (2) holds. Also, for every random moment t_0, independent of U_c^0, the distance between t_0 and the nearest preceding request (or the moment $t = 0$, if this request is absent) has distribution $\phi_1(t)$.

2.4. The assumptions on $G^c(\gamma(c))$ are the following:

(i) There exists a sequence of positive numbers $\{q(m)\}$, $m \in N$, such that for every $c \in C$, $\forall \gamma(c)$ with the length m, $G^c(\gamma(c)) < q(m)$.

(3) (ii) $$\sum_{m=1} m^2 v^m q(m) < \infty.$$

2.5. The initial condition W^0 is supposed to be independent of U^0.

3. Process of waiting times. The correctness of the service discipline

3.1. Let us define the waiting-time process W^t under a trivial initial condition. We imagine a fictitious request $(t, s, \gamma(c))$. The corresponding waiting time is $W^t[\gamma(c)]$. The vector $W^t = \{W^t[\gamma(c)]\}$, $c \in C$, $\gamma(c) \in \Gamma_c$ is a vector of fictitious waiting times. We need now to prove that all finite-dimensional distributions are proper that occur iff $\forall c$, $\forall \gamma(c) \in \Gamma_c$, $W^t[\gamma(c)]$ is finite almost certainly. The process of waiting times which corresponds to the initial conditions W_1^0 will be designated $W^t(W_1^0)$. It is easy to see (cf. [1], [2]) that if $W_1^0 = W^\tau$ then

$$W^{t+\tau} \overset{d}{=} W^t(W_1^0) \text{ (these processes have the same distributions).}$$

3.2. Before proving correctness we shall introduce some convenient notions. We fix a request $(t_0, s_0, \gamma(c_0))$ and give all definitions with respect to it. Consider an ordered set of vertices $c_1, \cdots, c_n$ and routes $\gamma(c_1), \cdots, \gamma(c_n)$. We say that they form a plane chain $\gamma(\bar{c}) = (\gamma(c_1), \cdots, \gamma(c_n))$ if $\gamma(c_i) \cap \gamma(c_{i+1})$, $i = 0, 1, \cdots, n-1$. The set of such chains of length n is designated $\Gamma_n(\bar{c})$, $\bar{c} = (c_1 \cdots c_n) \in C^n$. For convenience $c_i(\bar{c}) = c_i$, $\gamma_i(\gamma(\bar{c})) = \gamma(c_i)$.

Consider an ordered set of requests $(t_i, s_i, \gamma(c_i))$, $i = 1, \cdots, n$. We say that they form a space chain $\gamma(\bar{k}, \bar{c}, \bar{t})$, $\bar{k} = (k_1 \cdots k_n) \in N^n$, $\bar{c} \in C^n$, $\bar{t} = (t_1 \cdots t_n) \in R_+^n$, iff

(i) $t_{i-1} \geqq t_i$, $i = 1, \cdots, n$.

(ii) There are k_i requests on the vertex c_i in the interval $[t_{i-1}, t_i]$, $i = 1, \cdots, n$.

(iii) Their routes form a plane chain.

The set of such chains is denoted by $\Gamma_n(\bar{k}, \bar{c})$. For convenience $c_i(\bar{c}) = c_i$, $\gamma_i(\gamma(\bar{k}, \bar{c}, \bar{t})) = \gamma(c_i)$, $k_i(\bar{k}) = k_i$, $t_i(\bar{t}) = t_i$. The set of all requests, any of which can be included in some space chain, is called an influencing cone (of the request $(t_0, s_0, \gamma(c_0))$). Let $A_n(\bar{k}, \bar{c})$ be an event such that there exists $\gamma(\bar{k}, \bar{c}, \bar{t}) \in \Gamma_n(\bar{k}, \bar{c})$ and denote $A_n = \bigcup_{\bar{k},\bar{c}} A_n(\bar{k}, \bar{c})$.

Let A be the upper limit of A_n, $A = \bigcap_{n=1}^{\infty} \bigcup_{m \geqq n} A_m$.

3.3. To prove the correctness we show that the influencing cone is finite. This automatically implies the finiteness of waiting time and the correctness of service discipline. According to the Borel–Cantelli lemma $P(A) = 0$ if $\Sigma_{n=1} P(A_n) < \infty$. Thus the last inequality proves the correctness and we shall prove it now.

We have $P(A_n) \leqq \Sigma_{\bar{c} \in C^n} \Sigma_{\bar{k} \in N^n} P(A_n(\bar{k}, \bar{c}))$. Let $D_n(\bar{k}, \bar{c})$ be an event such that there exists a set of requests $(t_i, s_i, \gamma(c_i))$, $i = 1, \cdots, n$ satisfying conditions (i)–(ii) of the definition above. By definition, $P(A_n(\bar{k}, \bar{c})) = P(A_n(\bar{k}, \bar{c}) \mid D_n(\bar{k}, \bar{c})) \cdot P(D_n(\bar{k}, \bar{c}))$. The probability $P(A_n(\bar{k}, \bar{c}) \mid D_n(\bar{k}, \bar{c}))$ does not depend on $\bar{k} \in N^n$, which permits us to introduce one special construction. Consider the Cartesian product $N \times \Gamma$ and a random route Γ_n^c commencing from each vertex $(n, c) \in N \times \Gamma$. The probability $P(A_n(\bar{k}, \bar{c}) \mid D_n(\bar{k}, \bar{c}))$ is equal to the probability $P\{\exists \gamma(\bar{c}) \in \Gamma_n(\bar{c})\}$ that $\Gamma_1^{c_1}, \cdots, \Gamma_n^{c_n}$ form a plane chain $\gamma(\bar{c}) \in \Gamma_n(\bar{c})$, where $c_i(\bar{c}) = c_i$ (remember that all the definitions are made with respect to $(t_0, s_0, \gamma(c_0))$). The probability $P(D_n(\bar{k}, \bar{c}))$ does not depend on $\bar{c} \in C^n$. It is easy to see that this probability is smaller than $P\{\xi_n + \eta_{\bar{k}} < t_0\}$, where ξ_n is the sum of n random variables with d.f. $\phi_1(t)$ and $\eta_{\bar{k}}$ is the sum of $(k_1 + \cdots + k_n - n)$ random variables with d.f. $\phi(t)$. All these variables are mutually independent. Thus

$$P(A_n) \leqq \left[\sum_{\bar{c} \in C^n} P\{\exists \gamma(\bar{c}) \in \Gamma_n(\bar{c})\} \right] \left(\sum_{\bar{k} \in N^n} P\{\xi_n + \eta_{\bar{k}} < t_0\} \right).$$

Note that the sum in square brackets is the mean value of the random variable Γ_n, which is defined as a number of all chains of length n for our special model.

For the sum in round brackets we note that since

$$P\{\xi_n + \eta_{\bar{k}} < t_0\} \leqq P\{\xi_n < t_0\} \cdot P\{\eta_{\bar{k}} < t_0\},$$

this sum is smaller than

$$2^n \cdot P\{\xi_n < t_0\} \sum_{k=0}^{\infty} 2^k \sum_{\bar{k} = (k, 0, \cdots, 0)} P\{\eta_{\bar{k}} < t_0\}.$$

Lemma. The mean value of Γ_n is bounded by $|\gamma(c_0)| M^n$, where

$$M = \sum_{n=1} n(n+1) v^n q(n) < \infty.$$

Proof. $M\Gamma_n = \Sigma\, G^{c_1}(\gamma(c_1)) \cdots G^{c_n}(\gamma(c_n))$, where the sum is for all plane chains

$\gamma(\bar{c}) \in \Gamma_n(\bar{c})$ and $\bar{c} \in C^n$, $\gamma_i(\gamma(\bar{c})) = \gamma(c_i)$. The number of routes of length m_2 intersecting with a fixed route of length m_1 is smaller than

$$m_1 \cdot \sum_{r=0}^{m_2} \nu^r \nu^{m_2 - r} = m_1(m_2 + 1)\nu^{m_2}.$$

So

$$M\Gamma_n \leqq \sum_{m_1 \cdots m_n \in N} \sum_{\gamma(\bar{c}) \in \Gamma_n(\bar{c}), |\gamma(c_i)| = m_i} G^c(\gamma(c_1)) \cdots G^{c_n}(\gamma(c_n))$$

$$\leqq \sum_{m_1 \cdots m_n \in N} \sum_{\gamma(\bar{c}) \in \Gamma_n(\bar{c}), |\gamma(c_i)| = m_i} q(m_1) \cdots q(m_m)$$

$$\leqq |\gamma(c_0)| \sum_{m_1 \cdots m_n \in N} m_1(m_1 + 1)\nu^{m_1} q(m_1) \cdots m_n(m_n + 1)\nu^{m_n} q(m_n)$$

$$= |\gamma(c_0)| \cdot M^n.$$

To evaluate the sum in round brackets we shall prove the following.

Assertion. Let θ_n be a sequence of independent identically distributed positive random variables with $P\{\theta_n = 0\} = 0$. Then for every $q > 0$, $\exists N_0 \in N$ such that $\forall n \geqq N_0$

$$P\{\theta_1 + \cdots + \theta_n < t\} < q^n.$$

Proof. $\exists \Delta > 0 : P\{\theta_i < \Delta\} < q^2/4$. Taking $N = [2t/\Delta] + 1$ we have

$$\forall \geqq N_0 P\{\theta_1 + \cdots + \theta_n < t\} \leqq [P\{\theta_i < \Delta\}]^{n/2} \cdot 2^n \leqq q^n.$$

Let us choose $q > 0$ such that $q \cdot M < 1$. Then there exists such $N_0 \in N$ that for every $n \geqq N_0$ $P\{\xi_n < t_0\} < (\frac{1}{2}q)^n$ and for every $|\bar{k}| \geqq N_0$ $P\{\eta_{\bar{k}} < t_0\} < (\frac{1}{2})^{|\bar{k}|}$. Then for $n \geqq N_0$ we have $P(A_n) < (qM)^n$. Therefore the series $\Sigma P(A_n)$ is majorized by the convergent geometric progression, and this completes the proof of the correctness.

3.4. The main question for networks of this type is the existence of the limiting distributions of the process W^t. As we have indicated above, $W^{t+\tau} \stackrel{d}{=} W^t(W^\tau)$. Taking into account the monotonicity of the process with respect to the initial condition we have $W^t \stackrel{d}{\leqq} W^{t+\tau}$. Therefore the limiting distributions exist if $MW^t[\gamma(c)] < Q < \infty$ uniformly for $t \in R_+$, $c \in C$. The limiting process $W = d \cdot \lim_t W^t$ will be stationary in the sense that $W^t(W) \stackrel{d}{=} W$, and this is the consequence of the limiting properties. Note that the existence of the limiting behaviour of W^t also implies the existence of limiting distributions of the process $Q^t = \{Q^t_c\}$ $c \in C$ where Q^t_c is the length of the queue on the vertex c at the moment t.

Theorem. Let the arrival stream U^0 satisfy the restrictions of Section 2. Then there exists $A > 0$ such that the network with the arrival stream $U^0_A = [AT^c_n, S^c_n, \Gamma^c_n]$ has limiting distributions. One can see that if the distribution of arriving intervals or service times is exponential, the theorem can be formulated in the term of the load. The assertion of the theorem means that the dilatation of the service time or the contraction of the arrival time provides stability of the system. We can also notice that the theorem can be formulated for different service and arrival times for each vertex. Then the conditions (2.2) and (2.3) are assumed to hold uniformly for $c \in C$. Similar results can be obtained for other definitions of the term 'route' (but, of course, for our type of protocol).

4. A special case (Poisson stream)

4.1. In this section we consider one special model, which satisfies the restrictions of Section 2, but is essentially easier to investigate. The set of the vertices of the graph is $C \cong Z^1$ (one-dimensional lattice). All neighbouring vertices are linked by an edge. Only one type of route is considered from any vertex C — the route $\hat{\gamma}(c)$, consisting of two edges $(c, c+1)$ and $(c+1, c+2)$. Thus $G^c(\hat{\gamma}(c)) = 1$. The service times have exponential distribution with parameter μ, the interarrival times are exponentially distributed with parameter β. The load $\rho = \beta/\mu$.

Theorem. There exists $\rho_0 < 1$ (we can show that $\rho_0 > 1/10$) such that $\forall \rho \leqq \rho_0$ there exists a limiting process of waiting times.

Proof. As we have noticed earlier it is enough to show that $\forall c_0 \in \mathbb{Z}^1$, $\forall t_0$ for the waiting time of the request $(t_0, s_0, \gamma(c_0))$ we have $MW^{t_0}[\gamma(c_0)] < Q$. We shall introduce a new notion, which will be useful below. The space chain $\gamma(\bar{k}, \bar{c}, \bar{t}) \in \Gamma_n(\bar{k}, \bar{c})$ is called an essential space chain iff

(i) for $i = 0, \cdots, n-1$ the request $(t_i, s_i, \gamma(c_i))$ will be served immediately after the request $(t_{i+1}, s_{i+1}, \gamma(c_{i+1}))$ will leave the network.

(ii) The request $(t_n, s_n, \gamma(c_n))$ does not wait. In our special case $k_i = 1$, $i = 1, \cdots, n$ and $(c_i - c_{i+1}) \in A = \{-1, 0, 1\}$. The indicator of the event such that essential space chain passes through the vertices $c_0, c_0 + i_1, c_0 + i_1 + i_2, \cdots, c_0 + i_1 + \cdots + i_n$ where $i_j \in A$, $j = 1, \cdots, n$, is designated $I(i_1 \cdots i_n)$. For convenience we assume $t_0 = t$. Then

$$W^t[\hat{\gamma}(c_0)] = \sum_{n=1}^{\infty} \sum_{i_1 \cdots i_n \in A} W^t[\hat{\gamma}(c_0)] \cdot I(i_1 \cdots i_n)$$

$$\begin{aligned} MW^t[\hat{\gamma}(c_0)] \cdot I(i_1 \cdots i_n) &\leqq M(s_1 + \cdots + s_n) I(s_1 + \cdots + s_n > t - t_n) \\ &= \int_0^{\infty} \int_u^{\infty} \left(s \frac{(\mu s)^{n-1}}{(n-1)!} \exp(-\mu s) \right) \frac{(\beta u)^{n-1}}{(n-1)!} \exp(-\beta u) du \\ &= \frac{n}{\beta \mu^2} \int_0^{\infty} e^{-s} \left(1 + \cdots + \frac{s^{n-1}}{(n-1)!} \right) \Big|_{s = u/\rho} \frac{u^{n-1}}{(n-1)!} e^{-u} du \\ &\leqq \frac{n}{\beta \mu^2} \sum_{k=0}^{n-1} C_{n-1+k}^k \frac{1}{\rho^k} \left(\frac{\rho}{1+\rho} \right)^{n+k} \\ &\leqq \text{const. } n^2 \left[\rho \left(\frac{2}{1+\rho} \right)^2 \right]^n. \end{aligned}$$

The first inequality is the majorization of the probability of the essential space chain by the probability of the existence of the space chain of corresponding type. Thus if $\rho(2/(1+\rho))^2 < \frac{1}{3}$ (that is $\rho > 5 + 2\sqrt{6}$),

$$MW^t[\hat{\gamma}(c_0)] \leqq \sum_{n=1}^{\infty} 3^n \left[\rho \cdot \left(\frac{2}{1+\rho}\right)^2\right]^n < Q < \infty$$

uniformly for $t_0 = t \in R_+$.

5. Proof of the theorem

Consider a network of general type. Taking q such that $2q \cdot M < 1$ choose $T_0 \in R_+$ such that

(i) $\theta = \mu(T_0/2 - S_0) > 1$.

(ii) $\theta \cdot e^{-\theta} < q/2e$.

Since $\phi(t)$ and $\phi_1(t)$ are continuous at $t = 0$ there exists $A > 0$ such that

$$P\{A \cdot T_n^c < T_0\} = \phi(T_0/A) < (q/2)^2$$

and

$$P\{AT_1^c < T_0\} = \phi_1(T_0/A) < (q/2)^2.$$

We shall show that for the network with stream U_A^0 the limiting distributions exists. For convenience let $A = 1$ and consider the original network (thus we assume $\phi(T_0) < (q/2)^2$ and $\phi_1(T_0) < (q/2)^2$). Our aim is to evaluate $W^t[\gamma(c_0)]$ of the fictitious request $(t_0, s_0, \gamma(c_0))$. (We again consider $t = t_0$.) Let $I_n(\bar{k}, \bar{c})$ be an indicator of the event such that essential space chain is of the class $\Gamma_n(\bar{k}, \bar{c})$. Then

$$W^t[\gamma(c_0)] = \sum_{n\in N} \sum_{\bar{k}\in N^n} \sum_{\bar{c}\in C^n} W^t[\gamma(c_0)] \cdot I_n(\bar{k}, \bar{c}).$$

Consider the variable $\chi_i = |t_{i-1} - t_i|$, $i = 1, \cdots, n$, where $t_i = t_i(\bar{t})$ is taken from the essential space chain $\gamma(\bar{k}, \bar{c}, \bar{t})$. It is the sum of one random variable with distribution $\phi_1(t)$ and $(k_i - 1)$ random variables with distribution $\phi(t)$. Thus $t_0 - t_n$ is the sum of (modulo $\bar{k}$) independent variables.

Let $J(\bar{k}, \bar{c}, m)$, $m = 0, 1, \cdots, n$ be an indicator of the event that exactly m from all these $|\bar{k}|$ variables are greater than T_0. We denote $I\{s_1 + \cdots + s_n > m \cdot T_0\}$ the indicator of the event $\{s_1 + \cdots + s_n > m \cdot T_0\}$. Then we have

$$MW^t[\gamma(c_0)] \cdot I_n(\bar{k}, \bar{c}) = M \sum_{m=0}^{|\bar{k}|} W^t[\gamma(c_0)] I_n(\bar{k}, \bar{c}) J(\bar{k}, \bar{c}, m).$$

Notice that $W^t[\gamma(c_0)] \cdot I_n(\bar{k}, \bar{c}) > 0$, if the essential space chain is $\gamma(\bar{k}, \bar{c}, \bar{t})$, formed by the requests $(t_i, s_i, \gamma(c_i))$, $i = 1, \cdots, n$ for some $c_1, \cdots, c_n \in C$ and $t_1, \cdots, t_n, s_1, \cdots, s_n \in R_+$. It is easy to see that

$$W^t[\gamma(c_0)] \cdot I_n(\bar{k}, \bar{c}) \leqq (s_1 + \cdots + s_n) \cdot I_n(\bar{k}, \bar{c}) \cdot J(\bar{k}, \bar{c}, \bar{m}) \cdot I\{s_1 + \cdots + s_n > mT_0\}.$$

For the other side $I_n(\bar{k}, \bar{c}) \leqq \tilde{I}_n(\bar{k}, \bar{c})$, where $\tilde{I}_n(\bar{k}, \bar{c})$ is the indicator of the existence of an arbitrary space chain of corresponding type (not only essential). Thus we obtain

$$W^t[\gamma(c_0)]\cdot I_n(\bar{k},\bar{c})\cdot J(\bar{k},\bar{c},m)$$
$$\leqq (s_1+\cdots+s_n)I\{s_1+\cdots+s_n>mT_0\}J(\bar{k},\bar{c},m)\cdot\tilde{I}_n(\bar{k},\bar{c}).$$

The variables $(s_1+\cdots+s_n)I\{s_1+\cdots+s_n>mT_0\}$, $\tilde{I}_n(\bar{k},\bar{c})$ and $J(\bar{k},\bar{c},m)$ are independent. Since $M\tilde{I}_n(\bar{k},\bar{c})\leqq P\{\exists\gamma(\bar{c})\in\Gamma_n(\bar{c})\}$ and $MJ(\bar{k},\bar{c},m)\leqq(q/2)^m C^m_{|\bar{k}|}$ we obtain

$$MW^t[\gamma(c_0)]\cdot I_n(\bar{k},\bar{c})$$
$$\leqq P\{\exists\gamma(\bar{c})\in\Gamma_n(\bar{c})\}\cdot\sum_{m=0}^{|\bar{k}|}C^m_{|\bar{k}|}(q/2)^{|\bar{k}|-m}MI\{s_1+\cdots+s_n>mT_0\}\cdot(s_1+\cdots+s_n).$$

We divide the sum in the square brackets into two parts: the first sum with m changing from $m=0$ till $m=[n/2]$ and the second part with m changing from $m=[n/2]+1$ till $m=|\bar{k}|$. The first sum is smaller than $n\cdot Ms_1\cdot\Sigma_{m=0}^{n/2}2^{|\bar{k}|}(q/2)^{|\bar{k}|}=\text{const}\cdot n^2\cdot(q/2)^{|\bar{k}|}$. Now let us evaluate the second part. Notice that there exists a representation $s_i=s_i'+s_i''$, $i=1,\cdots,n$, where $s_i'\leqq S_0$ and $s_i''\leqq\theta_i$, θ_i being distributed exponentially with parameter μ. We have

$$P\{s_1+\cdots+s_n>T\}\leqq P\{s_1'+\cdots+s_n'>T-nS_0\}\leqq P\{\theta_1+\cdots+\theta_n>T-nS_0\}.$$

If $m>n/2$

$$M(s_1+\cdots+s_n)I\{s_1+\cdots+s_n>mT_0\}\leqq n\int_0^\infty(nS_0+T)\cdot\frac{(\mu T)^{n-1}}{(n-1)!}\cdot e^{-\mu T}dT$$
$$\leqq\text{const}\cdot n\cdot\left.\frac{y^ne^{-y}}{n!}\right|_{y=2m(T_0/2-S_0)}$$
$$\leqq\text{const}\cdot n\cdot\frac{(2m)^n}{n!}\cdot e^{-2m}\cdot(\theta e^{-\theta+1})^{2m}$$
$$\leqq\text{const}\cdot n^2\cdot(q/2)^m.$$

Therefore the second part is smaller than $\text{const}\cdot n^2\cdot(q/2)^{|\bar{k}|}$. Thus we obtain that $MW^t[\gamma(c_0)]\cdot I_n(\bar{k},\bar{c})$ is smaller than $\text{const}\cdot n^2\cdot(q/2)^{|\bar{k}|}$. This provides the following evaluation for $MW^t[\gamma(c_0)]$:

$$MW^t[\gamma(c_0)]\leqq\text{const.}\sum_{n=1}\ \sum_{\bar{k}\in N^n|\bar{k}|=l=n}\ \sum_{\bar{c}\in C^n}n^2\cdot P\{\exists\gamma(\bar{c})\in\Gamma_n(\bar{c})\}\left(\frac{q}{2}\right)^{|\bar{k}|}$$
$$=\text{const.}\sum_{n=1}M\Gamma_n\cdot\sum_{l=1}^{\infty}C^l_{n+l}\left(\frac{q}{2}\right)^l$$
$$\leqq\text{const.}\sum_{n=1}^{\infty}M^n(2q)^n<Q<\infty.$$

This bound does not depend on $t=t_0$, so we have proved the theorem.

References

[1] Borovkov, A. A. (1972) *Stochastic Processes in Queueing Theory*. Nauka, Moscow.

[2] Franken, P., König, D., Arndt, V. and Schmidt, V. (1981) *Queues and Point Processes*. Akademie-Verlag, Berlin.

[3] Kelbert, M. Y. and Sukhov, Y. M. (1985) Weak dependence of random field, describing the state of network at low transient streams (in Russian). *Probl. Pered. Inf.* **21**, 89–98.

[4] Kelly, F. P. (1986) Blocking probabilities in large circuit-switched networks. *Adv. Appl. Prob.* **18**, 473–505.

[5] Malyshev, V. A. and Ignatyuk, I. A. (1987) Locally interacting processes with noncompact set of values (in Russian). *Vest. Mosk. Univ.* (*Ser. Math.*) N2, 3–6.

[6] Mikhaylov, V. A. and Rybko, A. N. (1986) Ergodicity region of channel-switching networks (in Russian). *Probl. Pered. Inf.* **22**, 66–84.