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Loss Networks in Thermodynamic Limit

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1 Introduction

1.1 Abstract

The study deals with circuit switched loss networks (LN) in the *thermodynamic* limit. The model is the so-called *perturbed free loss network*. It consists of the superposition of a free LN and of a second LN, called the *perturbation*. In the free loss network a *fixed routing* is used and each route consists of only one link and different routes do not intersect. On the other hand, the perturbation can have a fixed, alternative or adaptive routing. Moreover, all (arrival) perturbation rates are proportional to some *perturbation parameter* $\epsilon > 0$. For sufficiently small $\epsilon > 0$, we prove that the limits

$$\lim_{\Lambda \nearrow \mathbb{R}^\nu} \lim_{t \to \infty} \mathbb{P}_t^{\Lambda,\epsilon}, \quad \lim_{t \to \infty} \lim_{\Lambda \nearrow \mathbb{R}^\nu} \mathbb{P}_t^{\Lambda,\epsilon}$$

of the *time correlation functions* exist and are equal. When the perturbations involve only fixed routing, the limiting measure is the Gibbs one. In general the situation is more complicated since LNs with alternative or adaptive routing are no longer reversible. To cope with these more complicated situations, new cluster expansions for the limiting measure are obtained. In particular, limiting semi-groups are shown to depend analytically on ϵ. Our approach to these problems is based on a diagram estimation technique, previously applied to handle the dynamics of some quantum systems.

1.2 General Presentation

The aim of this study is to analyze circuit switched loss networks (LN) with fixed, alternative and adaptive routing in the *thermodynamic limit*, i.e. when these systems become infinitely large. Loss networks with fixed routing are suitable models of cellular radio systems, while LN with adaptive routing give a fair representation of large telecommunications systems. We note that both fixed and alternative routings are particular cases of *adaptive routings*, defined in §6. On the other hand, LN provide a special class of non space homogeneous interacting particle systems (e.g., see [12]). In particular, LN can be considered as *spin systems*, with a spin taking usually more than two values. The interaction in these systems is a kind of *hard core interaction* in the terminology of statistical mechanics.

In finite *volumes* (or areas), LN with fixed routing are *reversible*. This means, in particular, that their stationary measure has a *product form*, which can be written explicitly. In fact, this measure is a Gibbs one and the study of the thermodynamic limit has an equivalent formulation in terms of Gibbs fields. But, in general, LN operating with alternative or adaptive routing are no longer reversible, so that their stationary measures do not have any tractable analytic expression.

In the thermodynamic limit, the stationary measure of loss networks (as well as the Gibbs measure) can be non-unique. This case corresponds to *phase transition*. An example of translation invariant LN in dimension two is described in [13] (see also the review [10] for further references). In some cases in dimension one, translation invariant LN with fixed routing have no phase transition [6]; when they are not translation invariant, a phase transition phenomenon can exist, and then there are exactly two extreme stationary measures [6]. At this moment, it is worth remarking that the question of uniqueness of the stationary measure for one-dimensional LN with adaptive routing is still open. This problem does also arise in similar translation invariant models encountered in statistical mechanics, where the Gibbs measure in dimension one is unique, but in dimension more than one there exists a phase transition at sufficiently low temperature (in the context of LN, the role of the temperature is played by the arrival rates). In [10], a one-dimensional LN with fixed routing has been explicitly solved .

Our contribution here is to construct a new *cluster* expansion allowing us to analyze, in the thermodynamic limit, the dynamics and the stationary measure of a large variety of LN with adaptive routing and sufficiently small arrival intensities. This expansion proves to be extremely useful to deal with this class of problems and relies on an original approach, which consists in a diagram representation of measures. Similar methods have been applied to the dynamics of some quantum dynamical systems [2, 3, 5, 7].

The paper is organized as follows: In the next four sections, LN with fixed routing are considered; we prove several results and estimates (concerning in particular perturbed Markov chains), which lead to the fundamental Theorem 3 of §4. Then in §6 we introduce LN with alternative and adaptive routing and we explain why the proposed method works also in this more complicated situation.

2 Loss Networks and Thermodynamic Approximation

Following [11], in order to describe loss networks, we consider $G=(V,L)$ a non-oriented connected graph, where V is the *set of vertices* and L is the *set of links*. In the sequel, V and L are assumed to be countable. We suppose also that each link $g=(v,v')\in L$ comprises $c(g)\in\mathbb{Z}_+$ circuits, i.e. each link g has the *capacity* $c(g)$. Let $\mathcal{R}$ be a *set of routes*, where each *route* is just a finite subset of L. A call on route R uses $n_R(g)\in\mathbb{Z}_+$ circuits from link g. Calls requesting route R arrive according to a Poisson stream with intensity $\lambda_R\geq 0$. These Poisson streams are assumed to be mutually independent. A call of route $R=\{g_1,\dots,g_{|R|}\}$ is blocked and lost if, on some link $g_i\in R, i=1,\dots,|R|$, there are less than $n_R(g_i)$ free circuits on link g_i. Otherwise the call is connected and occupies simultaneously $n_R(g_i)$ free circuits on g_i, for a duration exponentially distributed, with rate $\mu_R>0$. Holding and arrival processes are supposed to be independent, as well as the successive holding periods. Hence a loss network $\mathcal{N}$ can be described formally by the array

$$\mathcal{N}=\{G,\mathcal{R};\ \lambda,\mu;\ c,n\}.$$

Remark. As well as for interacting particle systems (see [12]), the question of existence and uniqueness of this stochastic process is not straightforward. This problem will be considered in Section 4 and we assume for the moment that this process is well defined.

Thermodynamic approximation. It is natural to ask the question of how to approximate (in some sense) this process by a simpler one, for example, by a finite Markov chain. To this end, we introduce a formal scheme of *thermodynamic limit* or *thermodynamic approximation* of infinite loss networks.

Let be given a sequence of loss networks $\mathcal{N}^{(i)}$, $i\geq 1$, where $G^{(i)}=(V^{(i)},L^{(i)})$ and (formally)

$$\mathcal{N}^{(i)}=\{G^{(i)},\mathcal{R}^{(i)};\ \lambda^{(i)},\mu^{(i)};\ c^{(i)},n^{(i)}\},$$

so that all sets $V^{(i)}$, $L^{(i)}$, $G^{(i)}$, $\mathcal{R}^{(i)}$ are finite with

$$V^{(i)}\subset V^{(i+1)},\quad \bigcup_i V^{(i)}=V,\qquad L^{(i)}\subset L^{(i+1)},\quad \bigcup_i L^{(i)}=L,$$

$$\mathcal{R}^{(i)}\subset\mathcal{R}^{(i+1)},\qquad \bigcup_i \mathcal{R}^{(i)}=\mathcal{R},$$

and, for all $j>i$,

$$\lambda^{(j)}{}_{|\mathcal{R}^{(i)}}=\lambda^{(i)},\qquad \mu^{(j)}{}_{|\mathcal{R}^{(i)}}=\mu^{(i)},$$
$$c^{(j)}{}_{|\mathcal{R}^{(i)}}=c^{(i)},\qquad n^{(j)}{}_{|\mathcal{R}^{(i)}}=n^{(i)}.$$

Then $\mathcal{N}$ will be referred to as the *thermodynamic limit* of the sequence of loss networks $\{\mathcal{N}^{(i)}, i\geq 1\}$. Whenever it is possible to associate each i with a volume Λ_i and to consider $\mathcal{N}^{(i)}$ as a localization of $\mathcal{N}$ in Λ_i (written later $\mathcal{N}_{\Lambda_i}$), one comes up with the standard notion of thermodynamic limit.

Markovian description of loss networks. An *admissible route configuration* for a loss network $\mathcal{N}$ is a function $\eta:\mathcal{R}\to\mathbb{Z}_+$, such that

$$\sum_{R\in\mathcal{R}:g\in R}\eta(R)n_R(g)\leq c(g)\,, \tag{1}$$

for all $g\in G$, where $\eta(R)\geq 0$ denotes the number of calls requiring route $R\in\mathcal{R}$. By definition, $\eta(R)=0$ means that there are no calls needing route R for the configuration η. Correspondingly, an *admissible route configuration in volume* Λ in a loss network $\mathcal{N}_\Lambda$ is a function $\eta^\Lambda: cal R_\Lambda\to\mathbb{Z}_+$ such that

$$\sum_{R\in\mathcal{R}_\Lambda:g\in R}\eta^\Lambda(R)n_R(g)\leq c(g),\ \ \forall g\in G_\Lambda\,. \tag{2}$$

Denote by $\mathcal{A}$ (resp. $\mathcal{A}_\Lambda$) the *set of all admissible route configurations* (resp. in volume Λ). Let

$$\eta_t=\{\eta_t(R),\ R\in\mathcal{R}\},\quad t\in\mathbb{R}_+,$$

respectively

$$\eta_t^\Lambda=\{\eta_t^\Lambda(R),\ R\in\mathcal{R}_\Lambda\},\quad t\in\mathbb{R}_+$$

be the Markov process describing $\mathcal{N}$ (resp. $\mathcal{N}_\Lambda$). Thus, for any finite volume Λ, η_t^Λ is a continuous time homogeneous Markov chain with finite state space $\mathcal{A}_\Lambda$.

Generators of finite loss networks. Let H_Λ be the generator of the Markov chain η_t^Λ. Then the probability $\mathbb{P}_t^\Lambda(\eta_\Lambda)$ of η_t^Λ of being in state η_Λ at time t is given by

$$\mathbb{P}_t^\Lambda(\eta_\Lambda) = \mathbb{P}_0^\Lambda \exp(t\, H_\Lambda)(\eta_\Lambda),$$

for any initial distribution $\mathbb{P}_0^\Lambda$.

Let $\mathcal{B}(\mathcal{A}_\Lambda)$ denote the finite dimensional space $\mathcal{B}(\mathcal{A}_\Lambda)$ of real functions $f : \mathcal{A} \to \mathbb{R}$. The generator H_Λ has the following local structure:

$$H_\Lambda = \sum_{R\in\mathcal{R}_\Lambda} (V_R^{\text{in}} + V_R^{\text{out}}), \tag{3}$$

where V_R^{in}, V_R^{out} are bounded linear operators : $\mathcal{B}(\mathcal{A}_\Lambda) \to \mathcal{B}(\mathcal{A}_\Lambda)$, corresponding respectively to arrivals and services of calls on route R. For each admissible route configuration η^Λ, the operator V_R^{in} transforms the δ-measure

$$\delta_{\eta^\Lambda}(\cdot) \equiv \delta(\cdot - \eta^\Lambda)$$

into the measure

$$\begin{cases} \lambda_R(\delta_{\eta^\Lambda+\delta_R} - \delta_{\eta^\Lambda}) & \text{if } \eta^\Lambda + \delta_R \in \mathcal{A}_\Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

and all other δ-measures into 0, where, by definition, we put

$$\delta_R(R') = \begin{cases} 1, & R = R', \\ 0, & R \neq R', \end{cases}$$

and

$$(\eta^\Lambda + \delta_R)(R') = \begin{cases} \eta^\Lambda(R') + 1 & \text{if } R = R', \\ \eta^\Lambda(R') & \text{otherwise.} \end{cases} \tag{5}$$

Analogously, V_R^{out} transforms $\delta_{\eta^\Lambda}(\cdot)$ into the measure

$$\begin{cases} \eta^\Lambda(R)\, \mu_R\, (\delta_{\eta^\Lambda-\delta_R} - \delta_{\eta^\Lambda}) & \text{if } \eta^\Lambda(R) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

and all other δ-measures into 0.

In other words, we have

$$V_R^{\text{in}} f(\eta^\Lambda) = \begin{cases} \lambda_R(f(\eta^\Lambda + \delta_R) - f(\eta^\Lambda)) & \text{if } \eta^\Lambda + \delta_R \in \mathcal{A}_\Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and

$$V_R^{\text{out}} f(\eta^\Lambda) = \begin{cases} \eta^\Lambda(R)\mu_R(f(\eta^\Lambda - \delta_R) - f(\eta^\Lambda)) & \text{if } \eta^\Lambda(R) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Time correlation functions. For any finite set $A \subset \mathcal{R}$ and any admissible route configuration $\eta \in \mathcal{A}$, the quantity $s_A = (s_R \equiv \eta(R) \geq 0,\ R \in A)$ is called a *(finite) route subconfiguration* in η. For a given s_A, we introduce the *time correlation function* (or simply *correlation function*) given by

$$\mathbb{P}_t(s_A) \equiv \mathbb{P}_t^{\mathcal{N}}(s_A) = \mathbb{P}(\eta_t(R) = s_R,\ R \in A), \quad t \in \mathbb{R}_+, \tag{9}$$

for some initial distribution $\mathbb{P}_0^{\mathcal{N}}$. Accordingly, the *time correlation function in volume* Λ is given by

$$\mathbb{P}_t^\Lambda(s_A) \equiv \mathbb{P}_t^{\Lambda,\mathcal{N}}(s_A) = \mathbb{P}(\eta_t^\Lambda(R) = s_R,\ R \in A), \quad t \in \mathbb{R}_+, \tag{10}$$

for some initial distribution $\mathbb{P}_0^{\Lambda,\mathcal{N}}$ and $A \subset \mathcal{R}_\Lambda$.

Remark. The Markov process η_t^Λ describing a loss network with fixed routing in volume Λ is reversible and its stationary measure is given by, after setting $\rho_R \equiv \frac{\lambda_R}{\mu_R}$,

$$\pi^\Lambda(\eta^\Lambda) \equiv \mathbb{P}_\infty^\Lambda(\eta^\Lambda) = \lim_{t\to\infty} \mathbb{P}_t^\Lambda(\eta^\Lambda) = Z_\Lambda^{-1} \prod_{R\in\mathcal{R}_\Lambda} \frac{(\rho_R)^{\eta^\Lambda(R)}}{(\eta^\Lambda(R))!}, \tag{11}$$

for each $\eta^\Lambda \in \mathcal{A}_\Lambda$, where Z_Λ is the normalizing constant (or *partition function*)

$$Z_\Lambda = \sum_{\eta^\Lambda\in\mathcal{A}_\Lambda} \prod_{R\in\mathcal{R}_\Lambda} \frac{(\rho_R)^{\eta^\Lambda(R)}}{(\eta^\Lambda(R))!}. \tag{12}$$

Therefore, when $t \to \infty$, the correlation function has a limit given by

$$\pi^\Lambda(s_A) \equiv \lim_{t\to\infty} \mathbb{P}_t^\Lambda(s_A) = Z_\Lambda^{-1} \left(\sum_{\substack{\eta^\Lambda\in\mathcal{A}_\Lambda,\\ \eta^\Lambda(R)=s_R,\, R\in A}} \prod_{R\in\mathcal{R}_\Lambda} \frac{(\rho_R)^{\eta^\Lambda(R)}}{(\eta^\Lambda(R))!} \right), \tag{13}$$

where $\pi^\Lambda(s_A)$ is the stationary probability of having a finite route subconfiguration s_A in a route configuration.

3 Loss Networks in $\mathbb{R}^\nu$

Here we introduce a class of loss networks which can be naturally localized in $\mathbb{R}^\nu$.

Let $\nu, d \in \mathbb{Z}_+$ be fixed. Denote by Γ_d the set of non-oriented connected graphs $G = (V, L)$ with a finite or countable number of vertices such that each vertex $v \in V$ has at most d adjacent links $\in L$. For fixed parameters $0 < D_1 < D_2 < \infty$, let $\Gamma_d^\nu(D_1, D_2)$ be a subset of Γ_d such that, for each $G = (V, L) \in \Gamma_d^\nu(D_1, D_2)$, there is a function $X : V \to \mathbb{R}^\nu$ satisfying the following properties: $(x_v \equiv X(v),\ v \in V)$

1. $\|x_v - x_{v'}\| \geq D_1$, if $v \neq v'$.
2. if $\|x_v - x_{v'}\| \geq D_2$ then $(v, v') \notin L$.

where $x_v \equiv X(v), \forall v \in V$ and, $\forall x = (x_1, \ldots, x_\nu) \in \mathbb{R}^\nu$,

$$\|x\| = \max_i |x_i| .$$

Thus $\Gamma_d^\nu(D_1, D_2)$ consists of graphs from Γ_d, for which there is a function X such that the set of points $\{x_v, v \in V\}$ has the *hard core* $D_1 > 0$ and that there are no links longer than D_2.

Example 1. The regular lattice $\mathbb{Z}^\nu$ belongs to $\Gamma_d^\nu(D_1, D_2)$, with $d = 2\nu$ and D_1, D_2 are constants subject to the inequalities $0 < D_1 < 1$, $D_2 > 1$.

Next we fix a graph $G = (V, L) \in \Gamma_d^\nu(D_1, D_2)$ together with a function X satisfying the above conditions. To define the thermodynamic limit in a convenient way, we also fix some vertex $v_0 \in V$, remarking that the limiting LN will not depend on the choice of v_0.

Graphs and boundary conditions in finite volumes. For a finite *volume* (an open bounded set) $\Lambda \subset \mathbb{R}^\nu$, we denote by $G_\Lambda = (V_\Lambda, L_\Lambda)$ the maximal connected component of the restriction of the graph G to the volume Λ containing x_{v_0}. Next, we chose the set of routes $\mathcal{R}_\Lambda$ of $\mathcal{N}$ as follows: $\mathcal{R}_\Lambda$ consists of routes $R = \{g_1, \ldots, g_{|R|}\}$ such that $g_i \in L_\Lambda$, for each $i = 1, \ldots, |R|$. In fact, this choice, for each Λ, of the set $\mathcal{R}_\Lambda$ corresponds to the choice of a boundary condition for a LN in a finite volume. Other (more general) boundary conditions will not be considered in this study, although the same methods could also be applied.

Thermodynamic limit of loss networks. A loss network $\mathcal{N}_\Lambda$ in volume Λ can be described by

$$\mathcal{N}_\Lambda = \{G_\Lambda, \mathcal{R}_\Lambda;\ \lambda^\Lambda, \mu^\Lambda;\ c^\Lambda, n^\Lambda\},$$

where λ^Λ, μ^Λ and c^Λ and n^Λ are given by the corresponding restrictions of λ, μ, c, n. Given a sequence of volumes $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_i \subset \ldots$, such that $x_{v_0} \in \Lambda_i, \forall i \geq 1$ and

$$\bigcup_{i=1}^{\infty} \Lambda_i = \mathbb{R}^\nu ,$$

we say that $\mathcal{N}$ is the *thermodynamic limit* of the sequence $\{\mathcal{N}_{\Lambda_i}, i \geq 1\}$.

In the sequel, we shall impose some technical conditions on these loss networks.

Condition 1. The function $c : L \to \mathbb{Z}_+$ is uniformly bounded:

$$\|c\|_\infty \equiv \sup_{g \in L} c(g) < \infty. \tag{14}$$

Condition 2. The arrival intensities are *uniformly bounded*:

$$\sup_{R \in \mathcal{R}} \lambda_R < \infty.$$

Condition 3. The routes are *uniformly bounded*: there exists a constant $D > 0$, such that $|R| \leq D$, for all $R \in \mathcal{R}$.

Condition 3*. The arrival intensities have an *exponential decay* with parameter $r > 0$:

$$\|\lambda\|_r \equiv \sup_{g \in L} \left(\sum_{k=1}^{\infty} e^{rk} \sum_{\substack{R: g \in R \\ |R| = k}} \lambda_R \right) < \infty. \tag{15}$$

Condition 4. The output intensities are *uniformly positive and bounded*:

$$\sup_{R \in \mathcal{R}} \mu_R < \infty, \qquad \mu_b \equiv \inf_{R \in \mathcal{R}} \mu_R > 0. \tag{16}$$

Clearly, Condition 3* follows from Conditions 2-3.

Routes. Formally, no further conditions need to be imposed on the routes $\mathcal{R}$, except Conditions 3 and 3*. But in realistic models it is natural to consider the following classes of routes: *connected* routes and *connected self-avoiding* routes. Connected route R has the form $\{g_1, \ldots, g_{|R|}\}$, $g_i \in L$, $i = 1, \ldots, |R|$, where the graph with links $g_i = (v_i, v_{i+1})$, $i = 1, \ldots, |R|$ and with vertices $\cup_{i=1}^{|R|+1} v_i$ is connected. We denote by $\mathcal{R}^c$ the set of connected routes in the LN and by $\mathcal{R}_g^c$, $g \in L$, the set of routes containing the link g. Connected self-avoiding route R have the form $\{g_1, \ldots, g_{|R|}\}$, $g_i = (v_i, v_{i+1}) \in L$, $i = 1, \ldots, |R|$, where the vertices $v_1, \ldots, v_{|R|}, v_{|R|+1}$ are distinct. Analogously, $\mathcal{R}^{sa}$ will denote the set of connected self-avoiding routes. Clearly, $\mathcal{R}^{sa} \subset \mathcal{R}^c$. Routings from $\mathcal{R}^c$ are used in models of cellular radio networks, while routes from $\mathcal{R}^{sa}$ rather apply to telephone networks.

4 Main Results

First we will consider the question of existence and uniqueness of the dynamics of a loss network in an infinite volume. Some general results about existence and uniqueness of the dynamics of interacting particle systems [12] will be used.

Let $\mathcal{N} = \{G, \mathcal{R};\ \lambda, \mu;\ c, n\}$. The state space of the Markov process η_t is the set $\mathcal{A}$ of all admissible route configurations. Condition 1 is supposed to hold. Let $N = \|c\|_\infty < \infty$ and let $S = \{0, 1, \ldots, N\}$ be a topological space endowed with the discrete topology. Then $S^{\mathcal{R}}$ is a compact metric space with the product topology. Since $\mathcal{A}$ is closed in $S^{\mathcal{R}}$, $\mathcal{A} \subset S^{\mathcal{R}}$ is also a compact metric space with the induced topology. Denote by $C(\mathcal{A})$ the set of continuous functions on $\mathcal{A}$, regarded as a Banach space with the norm

$$\|f\| = \sup_{\eta \in \mathcal{A}} \|f(\eta)\|.$$

For $f \in C(\mathcal{A})$ and $R \in \mathcal{R}$, let

$$\Delta_f(R) = \sup_{\substack{\eta, \eta' \in \mathcal{A}: \\ \eta(R') = \eta'(R'), \forall R' \neq R}} \left| f(\eta) - f(\eta') \right| .$$

This should be thought of as a measure of the extent the function f depends on the number of calls on route R. Define the set of functions

$$D(\mathcal{A}) = \left\{ f \in C(\mathcal{A}) : \sum_{R\in\mathcal{R}} \Delta_f(R) < \infty \right\}$$

which will play the role of a core for the generator of the Markov process η_t, $t \in \mathbb{R}_+$. Clearly, $D(\mathcal{A})$ is dense in $C(\mathcal{A})$.

For $\eta \in \mathcal{A}$, $R \in \mathcal{R}$, $m \in \mathbb{N}$, denote by

$$c_R(\eta, m) = \begin{cases} \lambda_R & \text{if } \eta(R) = m-1 \text{ and } \eta^{(m,R)} \in \mathcal{A}, \\ (m+1)\mu_R & \text{if } \eta(R) = m+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\eta^{(m,R)}(R) = m$ and $\eta^{(m,R)}(R') = \eta(R')$, for all $R' \neq R$.

Theorem 1. *Let $G \in \Gamma_d^v(D_1, D_2)$, $0 < D_1 < D_2 < \infty$. Assume condition 1 is satisfied for the loss network $\mathcal{N}$ and that*

$$\sup_{R\in\mathcal{R}} \max(\lambda_R, \mu_R) < \infty. \tag{17}$$

Then, for $f \in D(\mathcal{A})$, the series

$$Hf(\eta) = \sum_{R\in\mathcal{R}} \sum_{n\in\mathbb{N}} c_R(\eta, n) \left(f(\eta^{(n,R)}) - f(\eta) \right) \tag{18}$$

converges uniformly and defines a function in $C(\mathcal{A})$. Moreover, H is a Markov pregenerator and its closure $\overline{H}$ is a Markov generator of a Markov semigroup in $C(\mathcal{A})$ with the core $D(\mathcal{A})$.

The proof of Theorem 1 is identical to the proof of Proposition 3.2 in Chapter 1 of [12].

Corollary 2. *Under the conditions of Theorem 1, the dynamics of $\mathcal{N}$ (i.e. the Markov process η_t, $t \in \mathbb{R}_+$) is well defined and unique.*

Let

$$d_R = \sup_{\eta\in\mathcal{A}} \eta(R) \quad \forall R \in \mathcal{R}.$$

We say that two distinct routes $R, R' \in \mathcal{R}$ are *independent* if there is a route configuration $\eta \in \mathcal{A}$ such that $\eta(R') = d_R$ and $\eta(R) > 0$. For $R \in \mathcal{R}$, we denote by I_R the number of routes $R' \neq R \in \mathcal{R}$ which are not independent of R.

Let also

$$M = \sup_R \sum_{R'\neq R} c_R(R'), \tag{19}$$

where

$$c_R(R') = \begin{cases} \displaystyle\sup_{\substack{\eta_1,\eta_2\in\mathcal{A}:\\ \eta_1(\tilde{R})=\eta_2(\tilde{R}),\, \tilde{R}\neq R'}} \sup_{n\in\mathbb{N}} |c_R(\eta_1, n) - c_R(\eta_2, n)| & \text{if } R \neq R', \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $c_R(R')$ can take only the values λ_R or 0. More precisely, $c_R(R') = 0$ if R is independent of R' or if $\mathrm{supp}(R) \cap \mathrm{supp}(R') = \varnothing$; in all other cases, $c_R(R')$ is equal to λ_R, otherwise. Thus

$$M = \sup_{R\in\mathcal{R}} I_R \lambda_R .$$

Let

$$\varepsilon = \inf_{R\in\mathcal{R}} \inf_{\substack{\eta_1,\eta_2\in\mathcal{A}:\eta_1(R)\neq\eta_2(R),\\ \eta_1(R')=\eta_2(R'),\, R'\neq R}} \{c_R(\eta_1, \eta_2(R)) + c_R(\eta_2, \eta_1(R))\}.$$

It is easy to see that

$$\epsilon \geq \inf_{R\in\mathcal{R}} (\lambda_R + \mu_R).$$

Theorem 3. *Assume all conditions of Theorem 1 are satisfied (including inequality (17)) for $\mathcal{N}$. If*

$$\sup_{R\in\mathcal{R}} I_R \lambda_R < \epsilon, \tag{20}$$

then the process η_t is ergodic.

The proof of Theorem 3 mimics the proof of Theorem 4.1 in Chapter 1 of [12], since (20) is equivalent to the condition $M < \epsilon$.

Corollary 4. *Under the conditions of Theorem 3, the loss network $\mathcal{N}$ is ergodic.*

Corollary 5. *If the conditions of Theorem 3 hold and (17) is replaced by the inequality*

$$\sup_{R\in\mathcal{R}} I_R < \inf_{R\in\mathcal{R}} (\lambda_R + \mu_R), \tag{21}$$

then the process η_t is ergodic.

Remark. From corollary 5, it follows that, if conditions 1-4 are satisfied then, for λ_R, $R \in \mathcal{R}$ sufficiently small, $\mathcal{N}$ is ergodic.

4.1 Perturbing a Free Loss Network

A loss network

$$\mathcal{N}^f = \{G, \mathcal{R}^f;\ \lambda^f, \mu^f;\ c^f, n^f\} \tag{22}$$

is said to be *free* if each route consists of only one link, i.e. $R \in \mathcal{R}^f$ if $R = \{g\}$, for some $g \in L$. Moreover, in the sequel, we shall always suppose that different links correspond to different routes.

For two loss networks $\mathcal{N}_1, \mathcal{N}_2$ having the same graphs capacities, such that $\mathcal{R}_1 \cap \mathcal{R}_2 = \varnothing$, the sum $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ can be defined in a obvious way. Let

$$\mathcal{N}(\epsilon) = \{G, \mathcal{R};\ \epsilon_{\text{in}}\lambda, \epsilon_{\text{out}}\mu;\ c, n\}, \tag{23}$$

where $\epsilon = (\epsilon_{\text{in}}, \epsilon_{\text{out}}) \in \mathbb{R}_+^2$.

Definition 6. Let $\mathcal{N}^f$ and $\mathcal{N}^p$ denote respectively a free loss network and a loss network which has no one-link route. The loss network

$$\mathcal{N}_\epsilon^{pf} = \mathcal{N}^f + \mathcal{N}^p(\epsilon), \tag{24}$$

will be referred to as a *perturbed free loss network* with the *perturbation* $\mathcal{N}^p(\epsilon)$, where ϵ is the *perturbation parameter*, $\epsilon \equiv (\epsilon_{\text{in}}, \epsilon_{\text{out}}) \in \mathbb{R}_+^2$. The operator

$$H_\Lambda^f = \sum_{R\in\mathcal{R}_\Lambda^f} \underbrace{1\otimes\cdots\otimes h_R\otimes\cdots\otimes 1}_{} \tag{25}$$

is the generator of $\mathcal{N}_\Lambda^f$ in volume Λ, where h_R ($R = \{g\}$) denotes the generator of a finite Markov chain with state space

$$\left\{0, n_R(g), \ldots, \left[\frac{c(g)}{n_R(g)}\right] n_R(g)\right\}.$$

Also let us denote by $H_{\Lambda,\epsilon} \equiv H_{\Lambda,\epsilon}^{pf}$ the generator of a perturbed free loss network in volume Λ. Then

$$H_{\Lambda,\epsilon} = H_\Lambda^f + \epsilon_{\text{out}} V_\Lambda^{\text{out}} + \epsilon_{\text{in}} V_\Lambda^{\text{in}}, \tag{26}$$

where the generator $\epsilon_{\text{out}} V_\Lambda^{\text{out}}$ (resp. $\epsilon_{\text{in}} V_\Lambda^{\text{in}}$) describes the service mechanism (resp. the arrival process) of calls in $\mathcal{N}^p(\epsilon)$. Let

$$H_{\Lambda,0} \stackrel{\text{def}}{\equiv} H_\Lambda^f + \epsilon_{\text{out}} V_\Lambda^{\text{out}} \tag{27}$$

be the generator of the Markov process where the arrivals in $\mathcal{N}^p(\epsilon)$ are cut off. Hence

$$H_{\Lambda,\epsilon} = H_{\Lambda,0} + \epsilon_{\text{in}} V_\Lambda^{\text{in}}. \tag{28}$$

For a finite route subconfiguration s_A, $|A| < \infty$, we denote by $\mathbb{P}_t^{\Lambda,\epsilon}(s_A)$ (resp. $\mathbb{P}_t^{\Lambda,0}(s_A)$) the correlation functions of the Markov process with generator $H_{\Lambda,\epsilon}$ (resp. $H_{\Lambda,0}$) and by

$$\pi_\epsilon^\Lambda(s_A) \equiv \mathbb{P}_\infty^{\Lambda,\epsilon}(s_A) = \lim_{t\to\infty} \mathbb{P}_t^{\Lambda,\epsilon}(s_A) \tag{29}$$

the corresponding stationary probabilities.

In order to formulate our main result, we will introduce now the notion of *A-connected diagrams*. Let $R_i = (g_1^{(i)}, \ldots, g_{l_i}^{(i)})$, $1 \le i \le k$, $g_j^{(i)} \in L_\Lambda$, $1 \le j \le l_i$ be a given route and denote by

$$\tilde{R}_i = \bigcup_j g_j^{(i)}$$

the *support* of this route.

Definition 7 (*A*-**connected diagrams**) . A *diagram* D is a sequence $((R_1, t_1), \ldots, (R_k, t_k))$, where $R_i \in \mathcal{R}$, $t_i \in \mathbb{R}$, for $1 \le i \le k$. Let A be a set of routes. The diagram D is said to be *A-connected* if

$$\tilde{R}_i \cap \sigma_{i-1}(A) \neq \varnothing, \quad \forall i = 1, \ldots, k,$$

where

$$\sigma_{i-1}(A) = \sigma_0(A) \cup \tilde{R}_1 \cup \cdots \cup \tilde{R}_{i-1}, \quad \sigma_0(A) = \tilde{A} \stackrel{\text{def}}{\equiv} \bigcup_{R\in A} \tilde{R}.$$

Notation. Hereafter multiple integrals and summations will frequently occur. To render formulas of a reasonable size, we shall write typically

$$\mathbf{d}s_k = ds_1\, ds_2 \ldots ds_k,$$

and

$$\int_{[0,\infty)^n} F\, \mathbf{d}t_n = \int_0^\infty \cdots \int_0^\infty F dt_1 \ldots dt_n$$

and

$$\int_{\Delta_n^t} F(t, s_1 s_2 \ldots s_n)\, \mathbf{d}s_n = \int_0^t \int_0^{s_1} \cdots \int_0^{s_n} F(t, s_1 \ldots s_n)\, ds_1\, ds_2 \ldots ds_n$$

where, for example, $0 \le s_n \le \cdots \le s_1 \le t$

Theorem 8. *Let $G \in \Gamma_d^\nu(D_1, D_2)$, $0 < D_1 < D_2 < \infty$ and $\epsilon_{\text{out}} > 0$ be fixed. Suppose conditions 1, 2, 3 (or 3* with a parameter $r > 0$ sufficiently large) and 4 are satisfied for the networks $\mathcal{N}^f$ and $\mathcal{N}^p$. Then there are constants $\epsilon_0 > 0$, $C > 0$, such that, for all $\epsilon \equiv \epsilon_{\text{in}} \in [0, \epsilon_0]$, the Markov process η_t^ϵ, which describes $\mathcal{N}_\epsilon^{pf}$, is ergodic and, for each finite route subconfiguration $s_A = \{s_{R_1}, \ldots, s_{R_k}\}$, $R_i \in \mathcal{R}$, $1 \le i \le k < \infty$, the following limits exist and are equal:*

$$\lim_{t\to\infty} \lim_{\Lambda\nearrow\mathbb{R}^\nu} \mathbb{P}_t^{\Lambda,\epsilon}(s_A) = \lim_{\Lambda\nearrow\mathbb{R}^\nu} \lim_{t\to\infty} \mathbb{P}_t^{\Lambda,\epsilon}(s_A) = \pi_\epsilon(s_A). \tag{30}$$

Moreover,

$$\pi_\epsilon(s_A) = \sum_{k=0}^\infty \epsilon^n C_n(s_A), \tag{31}$$

$$C_n(s_A) = \sum_{\substack{R_i\in R,\\ 1\le i\le n}} \int_{[0,\infty)^n} \pi_0 V_{R_n}^{\text{in}} \exp(t_n H_0) \ldots V_{R_1}^{\text{in}} \exp(t_1 H_0)(s_A)\, \mathbf{d}t_n \tag{32}$$

where the summations and integrals are taken over all A-connected diagrams

$$D = ((R_1, t_1), \ldots, (R_n, t_n)).$$

Moreover the constants $C_n(s_A)$ are independent of ϵ and there exists a constant $C(s_A) > 0$, independent of n, such that

$$|C_n(s_A)| \le C^n C(s_A). \tag{33}$$

Thus the series (31) is convergent for $|\epsilon| \le \frac{1}{C}$ and $\pi_\epsilon(s_A)$ depends analytically on ϵ.

The proof is given in the next section.

Remark. The existence of the first limit in (30) follows from Theorem 3.

Remark. When the loss network has a fixed routing, the Markov process $\eta_t^{\Lambda,\epsilon}$ is reversible and

$$\lim_{t\to\infty} \mathbb{P}_t^{\Lambda,\epsilon}$$

is the Gibbs measure in Λ. Therefore, studying the second limit in (30) is equivalent to characterizing the corresponding Gibbs measure in the thermodynamic limit, so that one can use powerful tools of statistical mechanics. In particular, for the measure π_ϵ, one can construct various cluster expansions (e.g. see [16]). On the other hand, the study of the first limit in (30) is usually more difficult. In fact, we will construct a new cluster expansion, allowing us to control both limits simultaneously and the method works also in non-reversible situations (e.g. for LNs with alternative or adaptive routing), as will show the generalization of Theorem 8 in §6.

5 Proof of Theorem 8

We prove this theorem under condition 3, i.e. when the routes and the input intensities are uniformly bounded. The case where the input intensities have an exponential decrease (see condition 3*) needs some slight modifications of the proof which will be omitted.

One shall proceed along the following lines. First, using perturbation series (see (60), Appendix B) for each finite volume Λ and route subconfiguration $s_A \subset \mathcal{R}_\Lambda$, we represent $\mathbb{P}_t^{\Lambda,\epsilon}(s_A)$ as a sum of A-connected diagrams in Λ. Next, an exponential bound will be proved for each diagram, which in turn will yield a cluster bound for the sum of all A-connected diagrams. Here a crucial role will be played by the so-called universal cluster bound (see Appendix A). Similar cluster bounds have been used to handle the dynamics of some infinite quantum systems (see e.g. [4]). In particular, this will show that the limits

$$\lim_{\Lambda\nearrow\mathbb{R}^\nu}\lim_{t\to\infty}\mathbb{P}_t^{\Lambda,\epsilon}(s_A),\qquad \lim_{t\to\infty}\lim_{\Lambda\nearrow\mathbb{R}^\nu}\mathbb{P}_t^{\Lambda,\epsilon}(s_A)$$

exist and are equal, for any sufficiently small $\epsilon \geq 0$.

It will be convenient to write, up to a slight distortion in the notation, $V_\Lambda \equiv V_\Lambda^{\text{in}}$, $V_R \equiv V_R^{\text{in}}$ for all Λ, R. Now, let a finite route subconfiguration s_A and the initial measure P_0^Λ be fixed. Then (see (60), Appendix B)

$$\mathbb{P}_t^{\Lambda,\epsilon}(s_A) = \sum_{k=0}^{\infty}\epsilon^k \int_{\Delta_k^t} \mathcal{K}(t,k;\mathrm{s}_k)\,d\mathrm{s}_k\,, \tag{34}$$

where we have set

$$\mathcal{K}(t,k;\mathrm{s}_k) = P_0^\Lambda \exp\left(s_k H_{\Lambda,0}\right)V_\Lambda \exp\left((s_{k-1}-s_k)H_{\Lambda,0}\right)\ldots V_\Lambda \exp\left((t-s_1)H_{\Lambda,0}\right)(s_A)\,.$$

Hereafter, a *diagram* D will often mean a time-dependent k-uple, still denoted by

$$D = ((R_1,s_1),\ldots,(R_k,s_k))\,,\quad R_i\in\mathcal{R}\,,\quad \text{with } 0\le s_k\le\cdots\le s_1\le t\,,$$

and its cardinality $|D|$ is equal to k. The k-th term of series in (34) is equal to

$$\begin{aligned}&\int_{\Delta_k^t} P_0^\Lambda \exp\left(s_k H_{\Lambda,0}\right)V_\Lambda \exp\left((s_{k-1}-s_k)H_{\Lambda,0}\right)\ldots V_\Lambda \exp\left((t-s_1)H_{\Lambda,0}\right)(s_A)\,d\mathrm{s}_k\\ &\quad= \sum_{R_1\in\mathcal{R}_\Lambda}\cdots\sum_{R_k\in\mathcal{R}_\Lambda}\int_{\Delta_k^t} I(D,t)(s_A)\,d\mathrm{s}_k\\ &\quad= \sum_{R_1\in\mathcal{R}_\Lambda}\cdots\sum_{R_k\in\mathcal{R}_\Lambda}\mathcal{I}(\hat{D},t)(s_A)\,,\end{aligned} \tag{35}$$

where, setting $\widehat{D} = (R_1,\ldots,R_k)$, with $|\widehat{D}| \equiv k$,

$$\begin{aligned}&I(D,t)(s_A) \stackrel{\text{def}}{=}\\ &P_0^\Lambda \exp\left(s_k H_{\Lambda,0}\right)V_{R_k}\exp\left((s_{k-1}-s_k)H_{\Lambda,0}\right)\ldots V_{R_1}\exp\left((t-s_1)H_{\Lambda,0}\right)(s_A)\,,\end{aligned} \tag{36}$$

is called the *contribution of the diagram* D and

$$\mathcal{I}(\widehat{D},t)(s_A) \stackrel{\text{def}}{=} \int_{\Delta_k^t} I(D,t)(s_A)\,d\mathrm{s}_k\,.$$

For the sake of brevity, we shall often write

$$\sum_{|\widehat{D}|=k}\mathcal{I}(\widehat{D},t) = \sum_{R_1\in\mathcal{R}_\Lambda}\cdots\sum_{R_k\in\mathcal{R}_\Lambda}\mathcal{I}(\widehat{D},t). \tag{37}$$

where the $\sum_{D\subset\Lambda}$ is taken over all $\widehat{D} = (R_1,\ldots,R_k)$, such that $R_i \in L_\Lambda$, for $1 \le i \le k$. Using (34) one has

$$\mathbb{P}_t^{\Lambda,\epsilon}(s_A) = P_0^\Lambda\left(1+\sum_{k=1}^{\infty}\epsilon^k\sum_{\substack{\widehat{D}\subset\Lambda:\\|\widehat{D}|=k}}\mathcal{I}(\widehat{D},t)\right)(s_A). \tag{38}$$

To find $\mathbb{P}_t^{\Lambda,\epsilon}(s_A)$ one has to sum over all admissible route configurations $\eta^\Lambda \in \mathcal{A}_\Lambda$ containing the subconfiguration s_A. In other words,

$$\mathbb{P}_t^{\Lambda,\epsilon}(s_A) = \sum_{\substack{\eta^\Lambda\in\mathcal{A}_\Lambda:\\ \eta^\Lambda(R)=s_R,\ \forall R\in A}}\mathbb{P}_t^{\Lambda,\epsilon}(\eta^\Lambda). \tag{39}$$

Lemma 9. *Let a finite set $A \subseteq \mathcal{R}_\Lambda$ and a finite route subconfiguration s_A be fixed. If $D=((R_1,s_1),\ldots,(R_k,s_k))$, $\widetilde{R}_i \in \mathcal{R}_\Lambda$, is not an A-connected diagram, then*

$$\sum_{\substack{\eta^\Lambda\in\mathcal{A}_\Lambda:\\ \eta^\Lambda(R)=s_R,\,R\in A}}\mathbb{P}_0^\Lambda I(D,t)(\eta^\Lambda) = 0\,, \tag{40}$$

for all $0 \le s_k \le \cdots \le s_1 \le t$ and for each initial distribution $\mathbb{P}_0^\Lambda$.

Proof. From the definition of $I(D,t)(\eta^\Lambda)$ given in (36), two types of operators arise: V_R and $\exp\left(sH_{\Lambda,0}\right)$. Fist, we have

$$\delta_\eta \exp\left(sH_{\Lambda,0}\right) = \sum_{\eta'\in\mathcal{A}_\Lambda} T_s(\eta,\eta')\delta_{\eta'}\,,$$

where $T_s(\eta,\eta')$ is the s-time-transition probabilty to go from state η to state η'. Secondly, for each δ-measure δ_η, $\eta \in \mathcal{A}_\Lambda$, the operator V_R produces a positive measure $\lambda_R\delta_{\eta+\delta_R}$ and a negative measure $\lambda_R\delta_\eta$ on $\mathcal{A}_\Lambda$ (see Eq. (4) and (6)). These measures are concentrated on route configurations η and $\eta+\delta_R$, respectively, which differ only on route R, and $(\eta+\delta_R)(R) = \eta(R)+1$. Formally,

$$\delta_\eta V_R = \begin{cases}\lambda_R(\delta_{\eta^\Lambda+\delta_R}-\delta_{\eta^\Lambda}) & \text{if } \eta^\Lambda+\delta_R\in\mathcal{A}_\Lambda\,,\\ 0 & \text{otherwise.}\end{cases} \tag{41}$$

Both measures are conserved by the dynamics of $\exp(sH_{\Lambda,0})$, from the very definition of $H_{\Lambda,0}$.

For $R \in \mathcal{R}_\Lambda$, let

$$\mathcal{R}^p_{\partial R} = \left\{ R' \in \mathcal{R}^p_\Lambda : \mathrm{supp}(R') \cap \mathrm{supp}(R) \neq \varnothing,\ (L_\Lambda \setminus \mathrm{supp}(R)) \cap \mathrm{supp}(R') \neq \varnothing \right\}.$$

We first prove the following

Lemma 10. *Let $\Lambda \subset \mathbb{R}^\nu$ and some route $R \in \mathcal{R}_\Lambda$ be fixed. Then, for each admissible configuration $\eta = \eta_f + \eta_p \in \mathcal{A}_\Lambda$, $\eta_f \in \mathcal{A}^f_\Lambda$, $\eta_p \in \mathcal{A}^p_\Lambda$,*

$$\delta_\eta V_R \exp(sH_{\Lambda,0}) = \lambda_R \sum_{k\geq 0} \sum_{\eta^1_p} \cdots \sum_{\eta^k_p} \int_{\Delta^s_k} P(\eta_{p,R}, \overline{L}_k, \eta'_{p,R}) \Phi(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k) \tag{42}$$

with

$$(\nu^{R,+}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k) - \nu^{R,-}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)) \otimes \nu^R(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k),$$

for all $s \geq 0$, where:

(i) $\eta_{p,R} \equiv \eta_{|\mathcal{R}^p_{\partial R}}$, $\sum_{\eta^1_p} \cdots \sum_{\eta^k_p}$ is taken over all sequences of finite subconfigurations $\eta^0_p = \eta_{p,R}, \eta^1_p, \ldots, \eta^k_p = \eta'_{p,R}$ of routes in $\mathcal{R}^p_{\partial R}$, such that $\eta^{i+1}_p = \eta^i_p - \delta_{R(i)}$, where $R(i)$ is a route in $\mathcal{R}^p_{\partial R}$. The subconfiguration $\eta_{p,R}$ is a localization of η_p in $\mathcal{R}^p_{\partial R}$ and

$$\overline{L}_k = (\eta^0_p, r_0; \eta^1_p, r_1; \ldots; \eta^k_p, r_k), \quad r_0 \equiv 0.$$

(ii) The conditional probability measures

$$\nu^{R,+}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k), \quad \text{and} \quad \nu^{R,-}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$$

(for the process evolving with the generator $H_{\Lambda,0}$) are defined on route configurations in $\mathcal{R}_{\mathrm{supp}(R)}$, under the condition that $\eta_{p,R}(r)$, $0 \leq r \leq s$, satisfies

$$\eta_{p,R}(r) = \eta^i_p \quad \text{on the time interval } [r_i, r_{i+1}),\ \forall i \leq k-1,$$

the jumps taking place at the instants r_i, $0 \leq i \leq k-1$ and the initial measures being respectively $\delta_{\eta+\delta_R}$ and δ_η.

(iii) A similar definition holds for the measure $\nu^R(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$, which is defined for the route configurations in $\mathcal{R}_{L_\Lambda \setminus \mathrm{supp}(R)}$ under the same conditions.

(iv) $P(\eta_{p,R}, \overline{L}_k, \eta'_{p,R})$ is the probability density function of the occurrence

$$\eta_{p,R}(r) = \eta^i_p, \quad \forall 0 \leq r \leq s \text{ and } 0 \leq i \leq k-1,$$

according to the definition given above in (ii).

The proof follows directly from (41) and from the definition of $H_{\Lambda,0}$.

Remark. The following properties take place:

(i) The measures $\nu^{R,+}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ and $\nu^{R,-}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ on one hand, $\nu^R(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ on the other hand, have non-intersecting supports. Moreover,

(ii) $\nu^{R,+}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ and $\nu^{R,-}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ have equal masses. The number of summands in 42 is uniformly bounded by a constant $C < \infty$ since, conditions 1 and 3, the number of different sequences of route subconfigurations $\eta^0_p, \ldots, \eta^k_p$ in $\mathcal{R}^p_{\partial R}$ is uniformly bounded.

Continuing with the proof of Lemma 9, we note that, if the diagram D is not A-connected, then there exists R_i such that $\widetilde{R}_i \cap \sigma_{i-1}(A) = \varnothing$, so that, by (41) and Lemma 10, it is easy to establish by induction that the sum

$$\sum_{\substack{\eta^\Lambda \in \mathcal{A}_\Lambda: \\ \eta^\Lambda(R) = s_R,\, R \in A}}$$

gives the variation of the measures

$$\nu^{R_i,+}(\eta_{p,R_i}; \eta'_{p,R_i}, s_{i-1} - s_i|\overline{L}_k) \quad \text{and} \quad \nu^{R_i,-}(\eta_{p,R_i}; \eta'_{p,R_i}, s_{i-1} - s_i|\overline{L}_k),$$

respectively positive and negative, which both are defined on $\mathcal{R}_{\mathrm{supp} R_i}$ and have the same variation. As the stochastic process with generator $H_{\Lambda,0}$ conserves the mass of these measures, we obtain Lemma 9. □

5.1 Edges and Exponential Contribution of A-Connected Diagrams

We define the edges of an A-connected diagram $D = ((R_1, s_1), \ldots, (R_k, s_k))$ in the following way. For each R_i, $1 \leq i \leq k$, let $j < i$ be the smallest integer such that

$$\gamma(R_i) \cap \sigma_j(A) \neq \varnothing.$$

Then we connect the levels s_i and s_j with a vertical line. By definition, A corresponds to the level $s_0 \equiv 0$. If there is no such $j \geq 1$, then $\gamma(R_i) \cap A \neq \varnothing$ and we connect the level s_i to the level s_0, i.e. to A. Next we fix constants $\gamma > 0$, $K > 0$, and define the *exponential contribution of the edge $\psi_i = (s_i, s_j)$* by

$$\varphi_{K,\gamma}(\psi_i) = K \exp\{-\gamma(s_j - s_i)\}. \tag{43}$$

Then the *exponential contribution* $\varphi_{K,\gamma}(D)$ of the diagram D is defined by

$$\varphi_{K,\gamma}(D) = \prod_{i=1}^k \varphi_{K,\gamma}(\psi_i). \tag{44}$$

Lemma 11. *Let a volume $\Lambda \subset \mathbb{R}^\nu$ be fixed. Then there exist constants $K > 0$, $\gamma' > 0$ such that*

$$\left\| \nu^{R,+}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k) - \nu^{R,-}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k) \right\|_1 \leq K' \exp(-\gamma' s), \tag{45}$$

for all $R \in \mathcal{R}$, $s \geq 0$, $r_0 = 0 \leq r_1 \leq \cdots \leq r_k \leq s$ and for each sequence of admissible configurations $\eta^0_p = \eta_{p,R}, \eta^1_p, \ldots, \eta^k_p = \eta'_{p,R}$ of routes in $\mathcal{R}^p_{\partial R}$ such that $\eta^{i+1}_p = \eta^i_p + \delta_{R'(i)}$, for some route $R(i)$ from $\mathcal{R}^p_{\partial R}$, $i = 1, \ldots, k$, where $\|\cdot\|_1$ denotes the variation norm.

Proof. Let the route R be fixed. On the interval $[r_i, r_{i+1})$, the measures $\nu^{R,+}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ and $\nu^{R,-}(\eta_{p,R}; \eta'_{p,R}, s|\overline{L}_k)$ evolve according to the same finite irreducible Markov chain. The number of different Markov chains and the number of states for each of them are uniformly bounded for all R (see conditions 1,3). Thus, the bound (45) follows from standard results on the exponential convergence to steady state for an irreducible Markov chain, during each time interval $[r_i, r_{i+1})$. The parameters can be uniformly chosen, since the number of different sequences of route configurations $\eta_p^0, \dots, \eta_p^k$ is uniformly bounded and conditions 2,4 are satisfied. □

Lemma 12. *There exist positive constants K' and γ' such that, for each A-connected diagram $D = ((R_1, s_1), \dots, (R_k, s_k))$, $\widetilde{R}_i \in \mathcal{R}_\Lambda$, $I(D,t)$ can be estimated by*

$$|I(D,t)| \le C(A) \prod_{i=1}^{k} \varphi_{K',\gamma'}(\psi_i), \tag{46}$$

for all $\Lambda \subset \mathbb{R}^\nu$ and $t \ge 0$, where the constant $C(A) > 0$ is independent of Λ, k, and t.

Proof. This lemma can easily be proved along the same lines as Lemma 9, by using formula (42) and the probabilistic interpretation of the measures $\nu^{R,+}(\cdot)$, $\nu^{R,-}(\cdot)$ and $\nu^R(\cdot)$. □

The following lemma plays a crucial role.

Lemma 13. *For any $\gamma > 0$, $K > 0$,*

$$\sum_{\substack{D \text{ is } A\text{-connected} \\ |D|=k}} \int_{\Delta_k^t} \varphi_{K,\gamma}(D)\, \mathrm{d}s_k \le C^k, \tag{47}$$

for all $k \ge 1$, where $C = \frac{4K}{\gamma}$.

Proof. It is an immediate consequence of Lemma 17 in Appendix A, after choosing the function $g(t) = K \exp(-\gamma|t|)$. □

Lemma 14. *Let π_0, π_0^Λ be the stationary measures of the loss networks $\mathcal{N}^f$, $\mathcal{N}_\Lambda^f$, respectively. Then there exist positive constants K', γ', such that, for each A-connected diagram*

$$D = ((R_1, s_1), \dots, (R_k, s_k)),\ \widetilde{R}_i \in \mathcal{R}_\Lambda,$$

1.

$$|\pi_0 V_{R_n} \exp(t_n H_0) \dots V_{R_1} \exp(t_1 H_0)(s_A)| \le C(A) \prod_{i=1}^{k} \varphi_{K',\gamma'}(\psi_i), \tag{48}$$

for all $t \ge 0$, where the constant $C_1(A) > 0$ is independent of k, t and D.

2.

$$\left|\pi_0^\Lambda V_{R_n} \exp(t_n H_0^\Lambda) \dots V_{R_1} \exp(t_1 H_0^\Lambda)(s_A)\right| \le C_2(A) \prod_{i=1}^{k} \varphi_{K',\gamma'}(\psi_i) \tag{49}$$

for all $\Lambda \subset \mathbb{R}^\nu$ and $t \ge 0$, where the constant $C_2(A) > 0$ is independent of Λ, k, t and D.

3.

$$\left| \sum_{\substack{D \text{ is } A_{co} \\ |D|=k}} \int_{[0,\infty]^k} \pi_0 V_{R_k} \exp(t_k H_0) \dots V_{R_1} \exp(t_1 H_0)(s_A)\, \mathrm{d}t_k - \int_{\Delta_k^t} I(D,t)(s_A)\, \mathrm{d}s_k \right|$$
$$\le C_3(A) K' \exp(-\gamma' t), \tag{50}$$

for all $\Lambda \subset \mathbb{R}^\nu$ and $t \ge 0$, where A_{co} in the sum stands for A-connected and the constant $C_3(A) > 0$ is independent of Λ, k, t and D.

Proof. The bounds (48), (49) are particular cases of Lemma 12 and the bound (50) can be easily proved by using Lemma 13. See also the proof of (65) in Appendix B. □

The proof of theorem 8 now follows directly from Lemma 14.

6 Loss Networks with Adaptive Routing

Here we introduce loss networks operating with adaptive (in particular alternative) routing. It seems useful to give a general formal definition of these LNs, since (see e.g. [9] and references therein) there are of practical importance. In this section, we slightly modify the notation.

Let $\mathcal{R}$ be a set of routes, where a route is a nonordered pair of vertices $\{v, v'\}$. This means we do not distinguish the routes from v to v' and from v' to v. Each route $R = \{v, v'\}$ has a finite set of subroutes S_R, where any subroute $r \in S_R$ is a connected subgraph $G_R^r = (V_R^r, L_R^r) \subset G$, such that $v, v' \in V_R^r$, where V_R^r, L_R^r are respectively sets of vertices and links. The support of a route R is defined as by

$$\mathrm{supp}(R) \equiv \bigcup_{r \in S_R} \mathrm{supp}(r).$$

Upon arrival at time t, a call of route R chooses a subroute from $r \in S_R$ and will use $n_R^r(g) \in \mathbb{Z}_+$ circuits on link g. Calls requesting route $R = \{v, v'\}$ form a Poisson stream of intensity $\lambda_R \ge 0$ and all these Poisson streams are independent.
The sets $\mathcal{A}$ (resp. $\mathcal{A}_\Lambda$) still denote all *admissible route configurations* (resp. in volume Λ). Now, an admissible route configuration for $\mathcal{N}$ with adapting routing is a function

$$\eta : \mathcal{R} \times \mathcal{R} \to \mathbb{Z}_+, \quad \text{such that} \quad \sum_{\substack{R, r \in \mathcal{R}: \\ g \in \mathrm{supp}(R), r \in S_R}} \eta(R, r) n_R^r(g) \le c(g), \tag{51}$$

for all $g \in G$, where $\eta(R, r) \ge 0$ denotes the number of calls requesting route $R \in \mathcal{R}$ and choosing subroute $r \in S_R$.
Similarly an admissible route configuration in volume Λ is a function

$$\eta_\Lambda : \mathcal{R}_\Lambda \times \mathcal{R}_\Lambda \to \mathbb{Z}_+ \quad , \text{ such that} \quad \sum_{\substack{R, r \in \mathcal{R}_\Lambda: \\ g \in \mathrm{supp}(R), r \in S_R}} \eta_\Lambda(R, r) n_R^r(g) \le c(g) \tag{52}$$

for all $g \in G_\Lambda$, where $\eta_\Lambda(R, r) \ge 0$ denotes the number of calls requesting route $R \in \mathcal{R}_\Lambda$ and choosing subroute $r \in S_R$. By definition, $R \in \mathcal{R}_\Lambda$ means that $\mathrm{supp}(R) \subset G_\Lambda$.

A subroute is chosen according to the following procedure called *adaptive routing*. On the set $\mathcal{R} \times \mathcal{A}$, one defines two functions $\mathcal{K}.(\cdot)$, $\mathcal{E}.(\cdot)$. For each route R and each configuration η, $\mathcal{K}_R(\eta)$ is a finite sequence of subroutes from S_R and $\mathcal{E}_R(\eta)$ is the length of this sequence. The function $\mathcal{K}_R(\eta)$ will describe the set of possible alternative subroutes (the number of which is $\mathcal{E}_R(\eta)$) to connect an arriving call of route R at time t, when the current route configuration is η.

The mechanism is as follows: At time t, for a configuration η, an arriving call of route $R = \{v, v'\}$ tries to use subroutes from $\mathcal{K}_R(\eta)$, by turns, in the order specified by $\mathcal{K}_R(\eta)$. A call of route $R = \{v, v'\}$ can not use a subroute r if, on some link $g \in L^r_R$, there are less than $n^r_R(g)$ circuits free from this link. Otherwise, the call is connected and simultaneously holds $n^r_R(g)$ free circuits on link $g \in G^r_R$, and the holding time is exponentially distributed with rate $\mu^r_R > 0$ (function of R and r). A call is blocked and lost if all subroutes from $\mathcal{K}_R(\eta)$ are unworkable. As in the preceding sections, all random variables describing the arrival and service processes are supposed to be independent.

Remark. If $|S_R| \equiv 1$ and $\mathcal{E}_R \equiv 1$, for each $R \in \mathcal{R}$, then we get a LN with fixed routing. If the functions $\mathcal{E}_R(\eta)$ and $\mathcal{K}_R(\eta)$ do not depend on η, then we have the *alternative routing*. Thus fixed and alternative routings can be considered as particular cases of adaptive routing.

In the case of adaptive routing, the following conditions 3a, 3a*, 4a will replace conditions 3, 3*, 4 earlier introduced in §3.

Condition 3a. The routes are *uniformly bounded*: there exists a constant $D < \infty$ such that $|\text{supp}(R)| \le D$ for all $R \in \mathcal{R}$.

Condition 3a*. The input intensities have an *exponential decay*, with parameter $d > 0$:

$$\|\lambda\|_r \equiv \sup_{g \in L}\Big(\sum_{k=1}^{\infty} e^{dk} \sum_{\substack{R: g \in R \\ |R| = k}} \lambda_R\Big) < \infty. \tag{53}$$

Condition 4a. The output intensities are *uniformly bounded*:

$$\sup_{R \in \mathcal{R}} \mu_R < \infty, \qquad \mu_b \equiv \inf_{R \in \mathcal{R}} \inf_{r \in S_R} \mu^r_R > 0.$$

Definition 15. The function $\mathcal{K}.(\cdot)$ is said to be *local* (with parameter $D > 0$), if it depends only on links belonging to a D-neighbourhood $L_{N(R)}$ of supp(R), where

$$L_{N(R)} \stackrel{\text{def}}{=} \{g \in L : \text{dist}(g, \text{supp}(R)) \le D\}.$$

Condition 5. The function $\mathcal{K}.(\cdot)$ of a loss network with adaptive routing is *local* with some finite parameter D.

Let us consider a perturbed free loss network $\mathcal{N}_\epsilon = \mathcal{N}^f + \mathcal{N}^p(\epsilon)$, $\epsilon = (\epsilon_{\text{in}}, \epsilon_{\text{out}})$,

$$\mathcal{N}^p(\epsilon) = \{G, \mathcal{R};\ \epsilon_{\text{in}}\lambda, \epsilon_{\text{out}}\mu;\ \mathcal{K}.(\cdot), \mathcal{E}.(\cdot);\ c, n\},$$

where $\mathcal{N}^f$ is a free loss network with fixed routing and the perturbation $\mathcal{N}^p$ is a loss network with adaptive routing. Before stating a variant of theorem 8, we have to define, as in §4.1, the operators V_R, $R \in \mathcal{A}$ and H_0.

Let Λ and $\eta \in \mathcal{A}_\Lambda$ be fixed. The operator $V_R \stackrel{\text{def}}{=} V_R^{\text{in},p}$, supp$(R) \subset \mathcal{R}_\Lambda$, corresponds to call arrivals of route R. Suppose that $\mathcal{K}_R(\eta) = (r_1, \ldots, r_m)$, where $m = \mathcal{E}_R(\eta)$, $r_i \in S_R$ for $i = 1, \ldots, m$. Let $N_R(\eta)$ be the serial number of the subroute chosen to connect a call of route R. (In case of blocking, we put $N_R(\eta) = 0$). Then for each admissible route configuration

$$\eta : \mathcal{R}_\Lambda \times \mathcal{R}_\Lambda \to \mathbb{Z}_+,$$

V_R transforms the δ-measure

$$\delta_\eta(\cdot, \cdot) \equiv \delta((\cdot, \cdot) - \eta)$$

into the measure

$$\begin{cases} \lambda_R(\delta_{\eta + \delta_{R,r}} - \delta_\eta) & \text{if } r = r_i \text{ for } i = N_R(\eta), \\ 0 & \text{otherwise,} \end{cases}$$

and all other δ-measures into 0, where, by definition, we have put

$$(\eta + \delta_{R,r})(R', r') = \begin{cases} \eta(R', r') + 1 & \text{if } R = R' \text{ and } r = r', \\ \eta(R', r') & \text{otherwise.} \end{cases}$$

The generator H_0 could be defined in a similar manner.

Theorem 16. *Let $G \in \Gamma^v_d(D_1, D_2)$, $0 < D_1 < D_2 < \infty$ and ϵ_{out} is fixed. Suppose conditions 1, 2, 3a (or 3a* with a parameter $d > 0$ sufficiently large), 4a and 5 are satisfied for the networks $\mathcal{N}^f$ and $\mathcal{N}^p$. Then there are constants $\epsilon_0 > 0$, $C > 0$, such that, for all $\epsilon \equiv \epsilon_{\text{in}} \in [0, \epsilon_0]$, the Markov process η^ϵ_t, which describes $\mathcal{N}^{pf}_\epsilon$ is ergodic and, for each finite route subconfiguration $s_A = \{s_{R_1}, \ldots, s_{R_k}\}$, $R_i \in \mathcal{R}$, $1 \le i \le k < \infty$, the limits in (30) exist and are equal. Moreover, (31)and (32) hold, with the bound (33), for some constant $C(s_A) > 0$ independent of n.*

Proof. It mimics the proof of Theorem 8, since the the main arguments relied on the fact that the operator V_R was depending only on the support of the route R. But our definition of supp(R), as well as conditions 3a, 3a*, 4a and 5, ensure that the corresponding lemmas in the proof of Theorem 8 are still also valid in this case. □

Appendix A: a Universal Cluster Bound

Consider the class C_n of non-oriented graphs, such that each member G of the class be a $n-tree$, i.e. it is connected, has n edges and $n+1$ vertices, denoted by $v \in V = \{0, 1, \ldots, n\}$. Moreover, for all $t \ge 0$, we associate to any vertex $v \in V$, a real number $t_v \ge 0$ (called a v-*time*), where

$$t_0 = 0 \le t_1 \le \cdots \le t_n \le t.$$

Setting then $\bar{t} = (t_0, t_1, \ldots, t_n)$, the pair $(G, \bar{t})$ will be called a *t-diagram*. We always assume that the coordinate t_0 of vertex 0 is equal to 0. Unless otherwise mentioned, a n-uple explicitly written $\bar{t}$ will be subject to the above constraints and his components will be also called *coordinates*.

Let us fix some function $g : \mathbb{R} \to \mathbb{R}$. For a graph $G = (V, L) \in C_n$, the quantity

$$\prod_{\substack{l \in L,\\ l=(v,v')}} g(t_v - t_{v'})$$

is called the *contribution* of the diagram $(G, \bar{t})$.

Lemma 17. *Let $g : \mathbb{R} \to \mathbb{R}$ be an even function, Riemann-integrable on any compact set $K \subset \mathbb{R}$. Then*

$$\sum_{G \in C_n} \int_{\Delta_n^t} dt_1 \ldots dt_n \prod_{\substack{l \in G,\\ l=(v,v')}} g(t_v - t_{v'}) \leq 8^n \left(\|g\|_1^t\right)^n , \tag{54}$$

where $\|g\|_1^t \stackrel{\text{def}}{=} \int_0^t |g(s)|\, ds$.

Proof. It suffices to prove (54) for non-negative functions. Clearly, (54) is equivalent to

$$\sum_{G \in C_n} \int_{\Delta_n^t} dt_1 \ldots dt_n \prod_{\substack{l \in G,\\ l=(v,v')}} g(t_v - t_{v'}) \leq 4^n \{\int_{-t}^{t} g(r)\, dr\}^n . \tag{55}$$

In fact, we will prove the following inequality:

$$\delta^n \sum_{\substack{t_i \in \mathbb{Z}_\delta,\\ 0<t_1<\cdots<t_n<t}} \sum_{G \in C_n} \prod_{\substack{l \in G,\\ l=(v,v')}} g(t_v - t_{v'}) \leq 4^n \delta^n \sum_{\substack{r_1 \in \mathbb{Z}_\delta,\\ |r_1| \leq t,\ r_1 \neq 0}} \cdots \sum_{\substack{r_n \in \mathbb{Z}_\delta,\\ |r_n| \leq t,\ r_n \neq 0}} \prod_{i=1}^{n} g(r_i) , \tag{56}$$

for all $\delta > 0$, $n \geq 1$, where $\mathbb{Z}_\delta$ the one-dimensional δ-lattice, $\mathbb{Z}_\delta \stackrel{\text{def}}{=} \delta \cdot \mathbb{Z}$. Since both sides of (56) are approximations by Riemann sums of both sides of (55), it suffices to let $\delta \to 0+$ to obtain (55).

Notation. It will be convenient to introduce two types of vectors:

(i)

$$\bar{r} = (r_1, \ldots, r_n), \quad \text{with } r_i \in \mathbb{Z}_\delta,\ \ 0 < |r_n| \leq t,\ \ 1 \leq i \leq n , \tag{57}$$

noting that $\bar{r}$ is not necessarily positive;

(ii)

$$\bar{q} = (q_0, \ldots, q_n), \quad \text{with } q_i \geq 0,\ q_i \in \mathbb{Z}_+ ,\ \forall 1 \leq i \leq n \quad \text{and} \sum_{i=0}^{n} q_i = n . \tag{58}$$

When the components of a vector belong to $\mathbb{Z}_\delta$, it will be said to satisfy the δ-*condition*.

Sketch of proof. For each vector $\bar{r}$, an algorithm will be given, allowing us to produce at most 4^n different diagrams $(G, \bar{t})$, $G \in C_n$ and $t_i \in \mathbb{Z}_\delta$, for $i \geq 1$, each diagram having a contribution equal to $\prod_{i=1}^n g(r_i)$. More exactly, it will be shown that each diagram $(G, \bar{t})$ can be constructed from a vector $\bar{r}$ (see the above definition) and, hence, 55 will follow.

Let us fix $t > 0$, $\bar{r}$ and $\bar{q}$ satisfying respectively (57) and (58). The algorithm $A(t, \bar{r}, \bar{q})$ presented hereafter constructs recursively at most one diagram $(G, \bar{t})$, which will be denoted by $I(t, \bar{r}, \bar{q})$ with $n + 1$ vertices and n edges, such that $\bar{t}$ satisfies the δ-condition and the contribution of $I(t, \bar{r}, \bar{q})$ is equal to $\prod_{i=1}^n g(r_i)$. Moreover, it will appear that

$$\bigcup_{G \in C_n} \bigcup_{\substack{\bar{t}: t_i \in \mathbb{Z}_\delta,\\ t_0=0<t_1<\cdots<t_n \leq t}} (G, \bar{t}) = \bigcup_{\bar{r}, \bar{q}} I(t, \bar{r}, \bar{q}) . \tag{59}$$

Since the number of distinct vectors $\bar{q}$, with $q_0 > 0$ does not exceed 4^n, inequality (56) follows immediately from (59).

The algorithm consists of at most $2(n + 1)$ steps enumerated by the sequence

$$(0, 1), \ldots, (0, q_0 + 1), \ldots, (n, 1), \ldots, (n, q_n + 1) .$$

If $q_0 = 0$, then the algorithm stops at step $(0, 1)$ without constructing any diagram. Otherwise, it starts with a vertex called vertex 0 and $t_0 = 0$. At step $(0, 1)$, the vertex 0 is referred to as being *current*. Then, choosing the number r_1, one constructs, from the current vertex 0, an edge leading to a new vertex r_1. At this moment, one will say that the number r_1 was *used* at step $(0, 1)$. At the final stage of the algorithm, the new vertex will be labelled as vertex i, for some i, with $t_i = r_1$, and we proceed by induction.

Let $r_1, \ldots, r_a$ be the numbers already used and assume we are at step (i, j). Then the algorithm works as follows:

(1) At step (i, j), take the number r_{a+1} and, if $q_i \geq j$, construct an edge length r_{a+1} from the current vertex v; otherwise, change the current vertex from which to construct an edge of length r_{a+1} (see stages 2 and 3 of the algorithm).
(2) At step $(i, 1)$, choose one of the vertices produced beforehand, say v, call v the current vertex for steps $(i, 1), \ldots, (i, q_i + 1)$ and draw edges from v at each of these steps. The vertex v is said to have been *used* at step $(i, 1)$.
(3) The choice of v is rendered unique according to the following rule: Among all previously constructed vertices, not used at previous steps, v is the vertex having the smallest coordinate. If such vertex does not exist, then our algorithm stops and does not produce any diagram.
(4) If a vertex is constructed with a coordinate either negative, or strictly greater than t, or if it coincides with a previously constructed vertex, then the algorithm stops and does not produce any diagram.
(5) The algorithm *normally* stops at step $(n, q_n + 1)$.

When the algorithm stops *normally*, it produces a graph. Enumerating its vertices by increasing order of their coordinates, it is easy to see that we have in fact constructed a diagram $(G, \bar{t})$, $G \in C_n$, such that $\bar{t}$ satisfies the δ-condition and the contribution is equal to $\prod_{i=1}^n g(r_i)$.

So it has just been shown that the algorithm constructs at most 4^n different diagrams $(G, \bar{t})$, each giving a contribution $\prod_{i=1}^n g(r_i)$.

It remains to prove that each diagram $(G, \bar{t})$ can be constructed by the algorithm for some $(\bar{r}, \bar{q})$, and (59) will be established. To that end, we slightly modify the above construction.

First, set $q_i = 0$, $\forall 0 \le i \le n$, and call the vertex 0 with coordinate t_0 *current*. Then consider all the links coming out from this vertex and let q_0 be their number. Next, order these links and let r_i be the (signed) length of the ith one. These links are said to be *are used* and the corresponding vertices to be *are constructed*, except vertex 0, which is also said to be *used*. Then, among all the already constructed (but not used) vertices, choose the one having the smallest coordinate, call it *current*, consider all the links coming out from the current vertex and let q_1 be the number of these links. Next order these links and let r_{i+q_0} be the length of the ith link $l = (v, v')$, counted positively if $t_v < t_{v'}$ and negatively otherwise. Say again that these links *are used* and the corresponding vertices are *constructed*, except the current vertex which is *used*, and proceed by induction. The process stops when all the links of diagram $(G, \bar{t})$ are *used*.

It is easy to see that if one takes the vectors $\bar{r}, \bar{q}$ obtained by the above procedure, then the algorithm will construct exactly the diagram $I(t, \bar{r}, \bar{q})$.

Lemma 17 is proved. □

Appendix B: Perturbation Theory for Stationary Probabilities of a Markov Chain

Here we derive a useful formula for stationary probabilities of a perturbed Markov chain.

Let $\mathcal{L}_0$, $\mathcal{L}_\epsilon$, $\epsilon \in \mathbb{R}$ be continuous-time-homogeneous Markov chains, with the same countable state space $\mathcal{A}$, defined by their respective generators H_0 and $H_\epsilon = H_0 + \epsilon V$, $\epsilon \in \mathbb{R}$. We assume that the operators H_0, V are bounded in $l_1(\mathcal{A})$. Then the probability $\mathbb{P}_t^\epsilon(\alpha)$, for the Markov chain $\mathcal{L}_\epsilon$, to be in α at time t can be expressed by the following formula:

$$\mathbb{P}_t^\epsilon(\alpha) = \sum_{k=0}^{\infty} \epsilon^k \int_{\Delta_k^t} \mathbb{P}_0 \exp(s_k H_0) V \exp((s_{k-1} - s_k) H_0) \dots V \exp((t - s_1) H_0)(\alpha)\, \mathrm{d}s_k\,, \quad (60)$$

where P_0 is the initial distribution of $\mathcal{L}_\epsilon$ and

$$\Delta_k^t = \{(s_1, \dots, s_k) \in \mathbb{R}^k;\ 0 \le s_k \le \dots \le s_1 \le t\}.$$

This is a well known formula for the perturbation of a semi-group in a Banach space (see e.g. [6, Theorem 3.3.33]), which is easy to derive when H_0 and V are bounded. Indeed, we have

$$\mathbb{P}_t^\epsilon = P_0 \exp(t(H_0 + \epsilon V))\,,$$

and it is possible to define, for all t, the following operator, bounded in $l_1(\mathcal{A})$,

$$W(t) = \exp(t(H_0 + \epsilon V))\exp(-t H_0)\,.$$

Then

$$\frac{dW(t)}{dt} = \epsilon W(t)\exp(t H_0) V \exp(-t H_0)\,,$$

whence

$$W(t) = W(0) + \epsilon \int_0^t W(s)\exp(s H_0) V \exp(-s H_0)\, ds$$

or, equivalently,

$$\exp(t H_\epsilon) = \exp(t H_0) + \epsilon \int_0^t \exp(s H_\epsilon) V \exp((-s + t) H_0)\, ds\,. \quad (61)$$

Now (60) is obtained by iterating (61). The series (60) converges in l_1, since

$$\sum_{k=0}^{\infty} |\epsilon|^k \int_{\Delta_k^t} \| \exp(s_k H_0) V \exp((s_{k-1} - s_k) H_0) \dots V \exp((t - s_1) H_0) \|_1\, \mathrm{d}s_k$$

$$\le \sum_{k=0}^{\infty} |\epsilon|^k \int_{\Delta_k^t} \|V\|_1{}^k\, \mathrm{d}s_k \ \le\ \exp(t|\epsilon| \|V\|_1), \quad \forall t \in \mathbb{R}_+\,.$$

Theorem 18. *Let $\mathcal{L}_0$ and $\mathcal{L}_\epsilon \epsilon \in \mathbb{R}$ be irreducible time-continuous-Markov chains, with countable state space $\mathcal{A}$, defined by their respective generators H_0 and $H_\epsilon = H_0 + \epsilon V$, $\epsilon \in \mathbb{R}_+$. Assume also $\mathcal{L}_0$ is ergodic and that:*

(A) There exist positive constants C and δ such that, for a given initial distribution P_0,

$$\| P_0 \exp(s H_0) - \pi_0 \|_1 \le C \exp(-\delta s)\,; \quad (62)$$

(B) The operator V is bounded in $l_1(\mathcal{A})$.

Then there exist positive constants ϵ_0, C_1, , δ_1 such that, for each $\epsilon \in [0, \epsilon_0]$, the Markov chain $\mathcal{L}_\epsilon$ is ergodic and its stationary distribution is given by the convergent series

$$\pi_\epsilon(\alpha) = \sum_{k=0}^{\infty} \epsilon^k \mathcal{E}(k)(\alpha) = \lim_{t\to\infty} \mathbb{P}_t^\epsilon(\alpha) = \lim_{t\to\infty} \sum_{k=0}^{\infty} \epsilon^k \mathcal{F}(t, k)(\alpha)\,, \quad (63)$$

for any $\alpha \in \mathcal{A}$ and any initial distribution P_0 of $\mathcal{L}_\epsilon$, where we set

$$\mathcal{F}(t, k)(\alpha) \stackrel{\text{def}}{=} \int_{\Delta_k^t} \mathbb{P}_0 \exp(s_k H_0) V \exp((s_{k-1} - s_k) H_0) \dots V \exp((t - s_1) H_0)\, \mathrm{d}s_k(\alpha)\,,$$

and

$$\mathcal{E}(k)(\alpha) \stackrel{\text{def}}{=} \int_{[0,\infty]^k} \pi_0 V \exp(t_k H_0) \dots V \exp(t_1 H_0)\, \mathrm{d}t_k(\alpha) \le C_1^k\,.$$

Moreover,

$$\|\mathcal{E}(k)\|_1 \le C_1^k\,, \quad (64)$$

and

$$\|\mathcal{E}(k) - \mathcal{F}(t, k)\| \le C_1^k \exp(-\delta_1 t)\,, \quad \forall k \in \mathbb{Z}_+,\ t \in \mathbb{R}_+\,. \quad (65)$$

Remark. Conditions A and B in the above Theorem 18 are rather strong. It can be shown that they hold for Markov chains which are either finite or countable and satisfying Doeblin's condition. For our purpose, it suffices to consider *finite* Markov chains. It might be worth noting that, for countable Markov chains, more general results on the analyticity of stationary distributions can be proved by means of Lyapounov functions (see [8, 14, 17]).

Proof. First one shows that (62) yields

$$\|V\exp(sH_0)\|_1 \leq C\|V\|_1 \exp(-\delta s)\,, \quad \forall s \in \mathbb{R}_+\,. \tag{66}$$

For each measure $x \in l_1(\mathcal{A})$, one can represent xV as $(xV)^+ - (xV)^-$, where

$$(xV)^+(\alpha) = \max(0, xV) \quad \text{and} \quad (xV)^-(\alpha) = \max(0, -xV).$$

Then

$$\|xV\|_1 = \|(xV)^+\|_1 + \|(xV)^-\|_1\,, \|(xV)^+\|_1 = \|(xV)^-\|_1\,.$$

The last equality proceeds from the fact that V is a difference of generators. Also, as $\exp(H_0 s)$ conserves positive measures, we get

$$(xV)^+\exp(H_0 s)(\beta) = \sum_\alpha (xV)^+(\alpha)p^0_{\alpha\beta}(s) = \|(xV)^+\|_1\pi^0_\beta + \sum_\alpha (xV)^+(\alpha)(p^0_{\alpha\beta}(s) - \pi^0_\beta)\,, \tag{67}$$

where $p^0_{\alpha\beta}(s)$ is the probability to be in state β at time s, when the initial distribution P_0 is δ_α. Now (66) follows from (67). Indeed,

$$\|((xV)^+ - (xV)^-)\exp(H_0 s)\|_1 = \sum_\beta \left| \sum_\alpha \big((xV)^+(\alpha) - (xV)^-(\alpha)\big)\,(p^0_{\alpha\beta}(s) - \pi^0_\beta)\right| \leq C\|V\|_1\exp(-\delta s)\,, \tag{68}$$

for some constant $C > 0$. Then (62) and (66) yield

$$\|\pi_0 V\exp(t_k H_0)\ldots V\exp(t_1 H_0)\|_1 \leq C^k\|V\|_1{}^k\exp(-(t_1 + \cdots + t_k)\delta) \tag{69}$$

and

$$\|(P_0\exp(s_k H_0) - \pi_0)V\exp((s_{k-1} - s_k)H_0)\ldots V\exp((t - s_1)H_0)\|_1 \leq C^k\|V\|_1{}^k\exp(-t\delta)\,, \tag{70}$$

since $t = t - s_1 + \cdots + (s_{k-1} - s_k) + s_k$. Hence, we obtain

$$\|\mathcal{E}(k)\| \leq \frac{C^k\|V\|_1{}^k}{\delta^k} \tag{71}$$

and

$$\|\mathcal{E}(k) - \mathcal{F}(t,k)\|_1 \leq \int_{\Delta^t_k} \mathcal{G}(t,k;\mathrm{s}_k)\,\mathrm{d}\mathrm{s}_k + \int_{\widetilde{\Delta}^t_k} C^k\exp(-(t_1 + \cdots + t_k)\delta)\,\mathrm{d}\mathrm{t}_k \leq (C\|V\|_1)^k\exp(-\delta t)\frac{t^k}{k!} + \left(\frac{C\|V\|_1}{\delta - \delta_1}\right)^k\exp(-\delta_1 t)\,, \tag{72}$$

for all $t \geq 0$, where δ_1 is fixed, $0 < \delta_1 < \delta$, $\mathcal{G}(t,k;\mathrm{s}_k)$ denotes the left-hand side member of the inequality (70) and

$$\widetilde{\Delta}^t_k = \left\{(t_1,\ldots,t_k), t_i \geq 0, 1 \leq i \leq k\,;\, \sum_i t_i \geq t\right\}.$$

Now choosing

$$0 < \epsilon_0 < \frac{\delta - \delta_1}{C\|V\|_1}\,,$$

(63) and (65) follow from (71) and (72), for positive constants C_1 and δ_1, which can easily be found. Theorem 18 is proved. □

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