Multi-agent Model of the Price Flow Dynamics

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Extended Abstract for TGF'11

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1 Introduction

Social models with many participants (clients, drivers, agents, ...) are models where the collective behaviour of a large social group is derived from the individual behavior (psychology) of the individuals of this group. There are a lot of such models, for example, queuing models with large number of customers and queues.

We present here a new kind of models. One can consider them as one instrument (for example, a stock) market with many participants (called particles), having various activities. Particle initially at $x(0) \in R_+$ is the seller who wants to sell one stock for the price $x(0)$, which is higher than the existing price $\beta(0) = 0$. One should not be confused with negative prices. By simple shift of 0 one can get positive prices for sufficiently long time period. There are K groups of sellers characterized by their activity to move towards more realistic (that is existing now) price. Similarly, the buyers, situated initially on R_- would like to buy a stock for the price lower than $\beta(0)$.

The main result of the paper is the explicit formula for the asymptotic velocity of the boundary as the function of $2(K+L)$ parameters — densities and initial velocities. It appears to be continuous but not smooth in the points of some hypersurface. This kind of phase transition has very clear interpretation. At the points where derivatives do not exist the particles with smaller activities (velocities) cease to participate in the boundary movement — they are always behind the boundary, that is do not influence the market price $\beta(t)$.

It is important to say that real graphs of the price dependence on time look differently on different time scales. Our main formula corresponds to the time scale when the agent activities and densities do not change much during some time period.

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2 Model and the main result

Initial conditions At time $t = 0$ on the real axis there is a random configuration of particles, consisting of $(+)$-particles and $(-)$-particles. $(+)$-particles and $(-)$-particles differ also by the type: denote $I_+ = \{1, 2, \ldots, K\}$ the set of types of $(+)$-particles, and $I_- = \{1, 2, \ldots, L\}$ - the set of types of $(-)$-particles. Let

$$0 < x_{1,k} = x_{1,k}(0) < \cdots < x_{j,k} = x_{j,k}(0) < \cdots$$

be the initial configuration of particles of type $k \in I_+$, and

$$\cdots < y_{j,i} = y_{j,i}(0) < \cdots < y_{1,i} = y_{1,i}(0) < 0$$

be the initial configuration of particles of type $i \in I_-$, where the first index is the number of the particle in the configuration, the second index is its type. Thus all $(+)$-particles are situated on R_+ and all $(-)$-particles on R_-. Distances between neighbor particles of the same type denote by

$$\begin{aligned} x_{j,k} - x_{j-1,k} &= u_{j,k}^{(+)}, \qquad k \in I_+, \quad j = 1, 2, \ldots \\ y_{j-1,i} - y_{j,i} &= u_{j,i}^{(-)}, \qquad i \in I_-, \quad j = 1, 2, \ldots \end{aligned}$$

where we put $x_{0,k} = y_{0,i} = 0$. The random configurations corresponding to the particles of different types are assumed to be independent. The random distances between neighbor particles of the same type are also assumed to be independent, and moreover identically distributed, that is random variables $u_{j,i}^{(-)}, u_{j,k}^{(+)}$ are independent and their distribution depends only on the upper and second lower indices. Our technical assumption is that all these distribution are absolutely continuous and have finite means. Denote $\mu_i^{(-)} = Eu_{j,i}^{(-)}$, $\rho_i^{(-)} = \left(\mu_i^{(-)}\right)^{-1}, i \in I_-$, $\mu_k^{(+)} = Eu_{j,k}^{(+)}$, $\rho_k^{(+)} = \left(\mu_k^{(-)}\right)^{-1}, k \in I_+$.

Dynamics We assume that all $(+)$-particles of the type $k \in I_+$ move in the left direction with the same constant speed $v_k^{(+)}$, where $v_1^{(+)} < v_2^{(+)} < \ldots < v_K^{(+)} < 0$. The $(-)$-particles of type $i \in I_-$ move in the right direction with the same constant speed $v_i^{(-)}$, where $v_1^{(-)} > v_2^{(-)} > \ldots > v_L^{(-)} > 0$. If at some time t a $(+)$-particle and a $(-)$-particle are at the same point (we call this a collision or annihilation event), then both disappear. Collisions between particles of different types is the only interaction, otherwise they do not see each other. Thus, for example, at time t the $j-$th particle of type $k \in I_+$ could be at the point

$$x_{j,k}(t) = x_{j,k} + v_k^{(+)} t$$

if it will not collide with some $(-)$-particle before time t.

We define the boundary $\beta(t)$ between plus and minus phases to be the coordinate of the last collision which occured at some time $t' \leq t$. For $t = 0$ we put $\beta(0) = 0$. Thus the trajectories of the random process $\beta(t)$ are piecewise constant functions, we shall assume them continuous from the left. We shall prove the a.e. existence of the limit

$$W = \lim_{t \to \infty} \frac{\beta(t)}{t} \tag{1}$$

which we call the asymptotical speed of the boundary. However our main goal is explicit calculation of W.

Main result For any pair (J_-, J_+) of subsets , $J_- \subseteq I_-$, $J_+ \subseteq I_+$, define the number

$$V(J_-, J_+) = \frac{\sum_{i\in J_-} v_i^{(-)}\rho_i^{(-)} + \sum_{k\in J_+} v_k^{(+)}\rho_k^{(+)}}{\sum_{i\in J_-} \rho_i^{(-)} + \sum_{k\in J_+} \rho_k^{(+)}}, \qquad V(I_-, I_+) = V$$

The following condition is assumed

$$\{V(J_-, J_+) : \ J_- \neq \varnothing, \ J_+ = \varnothing\} \cap \{v_1^{(-)}, ..., v_L^{(-)}, v_1^{(+)}, \ldots, v_K^{(+)}\} = \varnothing \tag{2}$$

If the limit $W = \lim_{t\to\infty} \frac{\beta(t)}{t}$ exists a.e., we call it the asymptotic speed of the boundary.

Theorem 1

The asymptotic velocity of the boundary exists and is equal to

$$W = V(\{1, \ldots, L_1\}, \{1, \ldots, K_1\})$$

where

$$L_1 = \arg\max_{l\in I_-} V(\{1, \ldots, l\}, I_+) = \max\left\{l \in \{1, \ldots, L\} : \ v_l^{(-)} > V(\{1, \ldots, l\}, I_+)\right\},$$

$$K_1 = \arg\min_{k\in I_+} V(I_-, \{1, \ldots, k\}) = \max\left\{k \in \{1, \ldots, K\} : \ v_k^{(+)} < V(I_-, \{1, \ldots, k\})\right\}.$$

Now we will explain this result in more detail. It is always true that $v_K^{(+)} < 0 < v_L^{(-)}$ and there can be 3 possible ordering of the numbers $v_L^{(-)}, v_K^{(+)}, V$:

1. $v_K^{(+)} < V < v_L^{(-)}$. In this case

$$K_1 = K, \quad L_1 = L, \quad W = V.$$

2. If $v_L^{(-)} < V$ then $V > 0$ and $K_1 = K, \quad L_1 < L$. Moreover

$$W = V(\{1, \ldots, L_1\}, I_+) = \max_{l\in I_-} V(\{1, \ldots, l\}, I_+) \ > \ V \ > 0.$$

3. If $v_K^{(+)} > V$ then $V < 0$ and $K_1 < K, \quad L_1 = L$. Moreover

$$W = V(\{1, \ldots, L\}, \{1, \ldots, K_1\}) = \min_{k\in I_+} V(\{1, \ldots, L\}, \{1, \ldots, k\}) \ < \ V \ < \ 0.$$

Another scaling Normally the minimal difference between consecutive prices (a tick) is very small. Moreover one customer can have many units of commodities. That is why it is natural to consider scaled densities

$$\rho_j^{(+),\epsilon} = \epsilon^{-1}\rho_j^{(+)}, \quad \rho_j^{(-),\epsilon} = \epsilon^{-1}\rho_j^{(-)}$$

for some fixed constants $\rho_j^{(+)}, \rho_j^{(-)}$. Then also the phase boundary trajectory $\beta^{(\epsilon)}(t)$ will depend on ϵ. The results will look even more natural. Namely, there exists the following limit in probability

$$W^0 = \lim_{\epsilon\to 0} W^\epsilon = \lim_{\epsilon\to 0} \frac{\beta^{(\epsilon)}(t)}{t}.$$

3 Phase transition and method of proof

No phase transition if activities are the same The case $K = L = 1$, that is when the activities of (+)-particles are the same (and similarly for (−)-particles), is very simple. There is no phase transition in this case. The boundary velocity

$$w = \frac{v_1^{(+)}\rho_1^{(+)} + v_1^{(-)}\rho_1^{(-)}}{\rho_1^{(+)} + \rho_1^{(-)}} \tag{3}$$

depends analytically on the activities and densities. This is very easy to prove because the n-th collision time is given by the simple formula

$$t_n = \frac{x_n^{(+)}(0) - x_n^{(-)}(0)}{-v_1^{(+)} + v_1^{(-)}} \tag{4}$$

and n-th collision point is given by

$$x_n^{(+)}(0) + t_n v_1^{(+)} = x_n^{(-)}(0) + t_n v_1^{(-)}. \tag{5}$$

More complicated situation was considered in [3]. There the movement of (+)-particles has constant drift $v_1^{(+)} \neq 0$ but also jumps in both directions (and similarly for (−)-particles). In [3] *the order* of particles of the same type *can be changed* with time. There are no such simple formula as (4) and (5) in this case. The result is however the same as in (3).

Example of phase transition The phase transition appears already in case when $K = 2$, $L = 1$ and moreover the (−)-particles stand still, that is $v_i^{(-)} = 0$. Denote $v_i^{(+)} = v_i$, $\rho_i^{(+)} = \rho_i$, $i = 1, 2$. Consider the function

$$V_1(v_1, \rho_1) = \frac{\rho_1 v_1}{\rho_0 + \rho_1}\,.$$

It is the asymptotic speed of the boundary in the system where there is no (+)-particles of type 2 at all.

Then the asymptotic velocity is the function

$$W = V(v_1, v_2, \rho_1, \rho_2) = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_0 + \rho_1 + \rho_2}$$

if $v_2 < V_1$ and

$$W = V_1(v_1, \rho_1) = \frac{\rho_1 v_1}{\rho_0 + \rho_1}$$

if $v_2 > V_1$. We see that at the point $v_2 = V_1$ the function W is not differentiable in v_2.

Method of proof The proofs are based on the reduction of the considered process to Markov random walk in the orthant R_+^N with $N = KL$. Namely, denote $b_i^{(-)}(t)$ ($b_k^{(+)}(t)$) the coordinate of the extreme right (left), and still existing at time t, that is not annihilated at some time $t' \leq t$, (−)-particle of type $i \in I_-$ ((+)-particle of type $k \in I_+$).

Define the distances $d_{i,k}(t) = b_k^{(+)}(t) - b_i^{(-)}(t) \geq 0$, $i \in I_-, k \in I_+$. The trajectories of the random processes $b_i^{(-)}(t)$, $b_k^{(+)}(t)$, $d_{i,k}(t)$ are assumed left continuous, for any indices. Consider the random process $D(t) = (d_{i,k}(t), (i,k) \in I) \in R_+^N$, where $N = KL$. $D(t)$ is a Markov process due to our assumptions concerning initial distribution.

Let us describe the trajectories $D(t)$ in more detail. The coordinates $d_{m,n}(t)$ decrease linearly with the speeds $v_m^{(-)} - v_n^{(+)}$ correspondingly until one of the coordinates $d_{m,n}(t)$ becomes zero. Let $d_{i,k}(t_0) = 0$ at some time t_0. This means that $(-)$-particle of type i collided with $(+)$-particle of type k. Let them have numbers j and l correspondingly. Then the components of $D(t)$ become:

$$
\begin{aligned}
d_{i,k}(t_0+0) &= u_{j+1,i}^{(-)} + u_{l+1,k}^{(+)}\,, \\
d_{i,m}(t_0+0) - d_{i,m}(t_0) &= u_{j+1,i}^{(-)}, \qquad m \neq k\,, \\
d_{n,k}(t_0+0) - d_{n,k}(t_0) &= u_{l+1,k}^{(+)}, \qquad n \neq i\,,
\end{aligned}
$$

and other components will not change at all, that is do not have jumps. Note that the increments of the coordinates $d_{n,m}(t_0+0) - d_{n,m}(t_0)$ at the jump time do not depend on the history of the process before time t_0, as the random variables. $u_{j,i}^{(-)}$ $(u_{j,k}^{(+)})$ are independent and equally distributed for fixed type. Markov property follows from this. Absolute continuity of the distributions of random variables $u_{j,i}^{(-)}$, $u_{j,k}^{(+)}$ garanties that the events when more than one coordinate of $D(t)$ become zero, have zero probability.

Note that the distances $d_{i,k}(t)$, for any t, satisfy the following conservation laws

$$
d_{i,k}(t) + d_{n,m}(t) = d_{i,m}(t) + d_{n,k}(t)
$$

where $i \neq n$ and $k \neq m$. Thus the state space $\mathcal{D}$ of our Markov process can be given as the set of non-negative solutions of the system of $(L-1)(K-1)$ linear equations

$$
d_{1,1} + d_{n,m} = d_{1,m} + d_{n,1}
$$

where $n, m \neq 1$. It follows that the dimension of $\mathcal{D}$ equals $K + L - 1$.

To study such kind of random walks, in [1] the Euler scaling was used to construct a special dynamical systems on the faces of the orthant. Here we also use large time limit to construct a deterministic dynamical system. The main advantage of our case is that this dynamical system appears to be the simplest possible. It either hits the fixed point — the origin (this case corresponds to $K_1 = K$, $L_1 = L$), or escapes to infinity along some face of R_+^N.

It is of interest also that the same techniques was used in a completely different situation, see [2] for analysis of some neural networks. There similar reduction was used and the same kind of dynamical system appeared.

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