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Markov Processes and Related Fields

Null Recurrent String

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Abstract. A finite string $\alpha = a_1 a_2 \dots a_n$ is a sequence of symbols from some alphabet $R = \{1, 2, \dots, r\}$. We define its Markovian evolution by some transition probabilities, dependent only on the right-most symbol, of erasing this symbol or of substituting it by two other symbols. In the case that such chains are null recurrent, we get limit laws for the distribution of the length of the string, of its right-most symbol and of the number of symbols i in the string in the large time limit. Applications of these results are random walks on some non-commutative groups and queues with several customer types.

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1. Main Results

A finite string $\alpha = a_1 a_2 \dots a_n$ is a sequence of symbols from an alphabet $R = \{1, 2, \dots, r\}$. We shall always enumerate finite strings from left to right, starting with 1; then $n = n(\alpha) = |\alpha|$ is called the length of the string and a_n its right-most symbol. The set of all finite strings, including the empty one $\emptyset$, is denoted by $\mathcal{A}$. Concatenation of two strings $\alpha = a_1 \dots a_n$ and $\beta = b_1 \dots b_m$ is defined by $\alpha\beta = c_1 \dots c_{n+m}$, where $c_1 = a_1, \dots, c_n = a_n$, $c_{n+1} = b_1, \dots, c_{n+m} = b_m$.

It is useful to consider also semi-infinite strings. A semi-infinite string is an infinite sequence $\alpha = \dots y_{n-1} y_n$ of symbols from the alphabet with a specified enumeration. More exactly, semi-infinite strings are defined by pairs (n, α), where $n \in \mathbf{Z}$ is called the position of the particle (or the right-most end of the string) and α is the environment on the left of the particle (which is changed by the particle), i.e. a function $\alpha : (-\infty, n] \to R$, for any $n < \infty$. The set of all semi-infinite strings is denoted by $\mathcal{A}_\infty$. The concatenation $\rho\delta$ of the semi-infinite string ρ and the finite string δ is defined similarly.

The evolution of semi-infinite strings is defined by the following transition probabilities

$$q(\gamma, \delta) = \mathsf{P}\{\xi_{t+1} = \rho\delta \mid \xi_t = \rho\gamma\},$$

where ρ is a semi-infinite string, and γ and δ are finite with $n(\gamma) = d, n(\delta) \leq 2d$. Note that they do not depend on ρ. The parameter d characterises the "depth" of the interaction.

For finite strings, one should also specify the transition probabilities $p_{\alpha,\beta}$ from strings α of length less or equal to d.

Throughout the paper we shall assume that $d = 1$. Generalisation to the case $d > 1$ seems straightforward, but demands a lot of technical work. Then $q(a, \emptyset)$ is the probability of erasing the last symbol x of the environment and of subsequently moving to the left, $q(a, b)$ is the probability that the particle does not move but substitutes the right-most symbol of the environment by a, and $q(a, bc)$ is the probability of a jump to the right whilst substituting a by the two symbols bc. By $\mathcal{L}$ we denote the Markov chain on the state space $\mathcal{A}$ with transition probabilities $\{q(a, \emptyset), q(a, b), q(a, bc), q(\emptyset, a), q(\emptyset, \emptyset)\}_{a,b,c\in R}$, with $q(\emptyset, a), q(\emptyset, \emptyset)$ the transition probabilities for the empty string to jump to a or to $\emptyset$ respectively. By ξ_t we denote the state of the Markov chain $\mathcal{L}$ at time t.

To avoid notational complications, we will always assume that all $q(\cdot, \cdot)$ are positive. Let us remind (see [5]) that a necessary and sufficient condition for null-recurrence of $\mathcal{L}$ is

$$\lambda = 1, \tag{1.1}$$

where λ is the maximal eigenvalue of the $r \times r$-matrix A defined by

$$A_{ab} = q(a, b) + \sum_c (q(a, bc) + q(a, cb)).$$

Let $e = (e_1, \dots, e_r)$ be the eigenvector corresponding to λ. Define a Lyapunov function

$$f(\alpha) = \sum_{a=1}^{r} e_a n_a, \tag{1.2}$$

where n_a is the number of symbols a in the string α. One can easily check that

$$(f(\xi_{t+1}) \mid \xi_t) = f(\xi_t), \text{ if } \xi_t \neq \emptyset.$$

To avoid unnecessary complications we consider an even simpler model defined by

$$\begin{aligned} p_{\rho a, \rho ab} &= \mathsf{P}\{\xi_{t+1} = \rho ab \mid \xi_t = \rho a\} = q_{ab}, \\ p_{\rho a, \rho} &= \mathsf{P}\{\xi_{t+1} = \rho \mid \xi_t = \rho a\} = q_a. \end{aligned}$$

We use the special symbol $*$ to denote an empty left end of the string or simply the empty string. Put $q_* = 0$.

It is interesting to remark, that the null-recurrence condition can also be written in the following form. Consider two matrices $Q = \{q_{ab}\}$ and $\widehat{Q} = \{q_a \delta_{ab}\}$ (a diagonal matrix). Then (1.1) becomes the condition that there exists a positive vector $\pi = \{\pi_a\}$, such that

$$\pi Q = \pi \widehat{Q}. \tag{1.3}$$

This can be interpreted in the following way. For some distribution of the last (right-most) symbol of the string, the mean drift of the particle (i.e. of $n(t)$, the length of the string at time t) is zero.

Hereafter in this section we will formulate our main results and we will prove these in the following sections. Our proofs are a mixture of purely probabilistic methods and complex variable methods.

Many authors considered random walks on free and similar groups. In the cases considered, only transient chains appeared (see [12] and references therein). Transient cases were covered in our previous papers in a more general situation (see [3], [5], see also review [4]). However, for the null recurrent case, no results existed in the literature until now.

Other applications are queueing models with several customer types.

1.1. Stabilisation law

Let $n(t) = |\xi_t|$ be the length of the string at time t and $\xi_t = a_1(t) \dots a_{n(t)}(t)$ the string itself at time t. Let

$$\mathrm{Last}_k(\xi_t) = a_{n(t)-k+1}(t) \dots a_{n(t)}(t), \quad k \le n(t),$$

be the right-most substring of ξ_t of length k at time t. By $\xi_t(k), k \le n(t)$ we denote $a_k(t)$.

Theorem 1.1. *Let δ be a string of length $k = |\delta|$. Then there exists a "limiting" probability $p_{\mathrm{Last}}(\delta)$, such that for any initial state $\beta \in \mathcal{A}$ of the Markov chain $\mathcal{L}$*

$$\mathsf{P}\{n(t) \ge k, \mathrm{Last}_k(\xi_t) = \delta \mid \xi_0 = \beta\} \to p_{\mathrm{Last}}(\delta),$$

as $t \to \infty$. This convergence is uniform in the set of all initial strings $\beta \in \mathcal{A}$, i.e. there exists $\psi_t(\delta) \to 0$ as $t \to \infty$, such that for any $\beta \in \mathcal{A}$

$$\big|\mathsf{P}\{n(t) \ge k, \mathrm{Last}_k(\xi_t) = \delta \mid \xi_0 = \beta\} - p_{\mathrm{Last}}(\delta)\big| \le \psi_t(\delta).$$

1.2. Mixing property

Theorem 1.2. *Let $p_t(a_1 \dots a_n) = \mathsf{P}\{\xi_t = a_1 \dots a_n \mid \xi_0 = *\}$ and*

$$\pi_{t,n}(a_1 \dots a_n) = \frac{1}{Z_{t,n}} p_t(a_1 \dots a_n),$$

where $Z_{t,n} = \mathsf{P}\{|\xi_t| = n \mid \xi_0 = *\}$. *Let* $\{\sigma_1, \ldots, \sigma_n\}$ *be random variables with values in* R *and with distribution* $\mathsf{P}\{\sigma_1 = a_1, \ldots, \sigma_k = a_n\} = \pi_{t,n}(a_1 \ldots a_n)$. *Let* $\mathcal{F}_k$ *be the* σ*-algebra on our probability space generated by the random variables* $\{\sigma_1, \ldots, \sigma_k\}$ *and* $\mathcal{F}^k$ *be the* σ*-algebra generated by the random variables* $\{\sigma_k, \ldots, \sigma_n\}$. *Then there exist* $C_1 > 0, 0 < C_2 < 1$, *such that for any* $0 \leq k < m \leq n$ *and any events* $A \in \mathcal{F}_k, B \in \mathcal{F}^m$

$$|\mathsf{P}(B \mid A) - \mathsf{P}(B)| < C_1 C_2^{m-k},$$

with the constants C_1, C_2 *not depending on* t, n.

1.3. Law of Large Numbers

Let $n_a(t)$ be the number of symbols a in the current string $\xi(t)$.

Theorem 1.3. *There exist positive* n_a *such that for any initial condition*

$$\frac{n_a(t)}{n(t)} \xrightarrow{\mathsf{P}} n_a, \text{ as } t \to \infty.$$

1.4. Central Limit Theorem

Theorem 1.4. *For some* $\sigma > 0$

$$\frac{n(t)}{\sigma\sqrt{t}} \xrightarrow{\mathcal{D}} |w|, \text{ as } t \to \infty,$$

in distribution, where w *is a normally distributed random variable with parameters* $(0, 1)$.

The next theorem follows as a corollary.

Theorem 1.5. *Let* $\vec{n}(t) = (n_1(t), \ldots, n_r(t))$. *Then the following limit exists,*

$$\frac{\vec{n}(t)}{\sqrt{t}} \xrightarrow{\mathcal{D}} |w|\vec{c}, \text{ as } t \to \infty,$$

where w *is normally distributed and the vector* $\vec{c}$ *is a constant vector.*

This result follows easily from the two previous theorems. But we will also give an independent analytic proof, which is of independent interest.

2. Proofs

First we will prove some auxiliary results.

Lemma 2.1. *Let*

$$p_{a_1\dots a_n}(z)=\sum_{t=0}^{\infty} z^t p_t(a_1\dots a_n).$$

Then for all $n>0$ *and all* $a_1,\dots,a_n$*, we have*

$$p_{a_1\dots a_n}(z)=z^n\varphi_*(z)\,q_{*a_1}\,\varphi_{a_1}(z)\,q_{a_1a_2}\,\varphi_{a_2}(z)\dots q_{a_{n-1}a_n}\,\varphi_{a_n}(z),$$

where

$$\begin{aligned}\varphi_a(z) &= \sum_{t\ge 0} z^t\varphi_a^t,\ \ \varphi_a^t=\mathsf{P}\{\xi_t=a,|\xi_k|\ge 1 \text{ for } k\le t\mid \xi_0=a\},\ \text{for } a\in R,\\ \varphi_*(z) &= \sum_{t\ge 0} z^t\mathsf{P}\{\xi_t=*\mid \xi_0=*\}.\end{aligned}$$

The functions $\{\varphi_a(z)\}_{a\in R\cup\{*\}}$ *satisfy the following system of equations*

$$\varphi_a(z)=1+z^2\sum_{b\in R} q_{ab}q_a\varphi_b(z)\varphi_a(z). \tag{2.1}$$

Proof. From the Markov property of the process ξ_t we have

$$\begin{aligned}p_t(a_1\dots a_n) &= \sum_{t_1\ge 0} p_{t_1}(a_1\dots a_{n-1})q_{a_{n-1}a_n} \\ &\times\mathsf{P}\{\xi_t=a_1\dots a_n,\ |\xi_k|\ge n \text{ for } k \text{ with } t\ge k\ge t_1+1\mid \xi_{t_1+1}=a_1\dots a_n\}.\end{aligned} \tag{2.2}$$

The third term in the right-hand side of (2.2) does not depend on $a_1\dots a_{n-1}$. So we can put

$$\begin{aligned}\varphi_{a_n}^{t-t_1-1} &= \mathsf{P}\{\xi_t=a_1\dots a_n,|\xi_k|\ge n \text{ for } k \\ &\qquad \text{with } t\ge k\ge t_1+1\mid \xi_{t_1+1}=a_1\dots a_n\}\\ &= \mathsf{P}\{\xi_{t-t_1-1}=a_n,|\xi_k|\ge 1 \text{ for } k\le t-t_1-1,\mid \xi_0=a_n\}.\end{aligned}$$

We have

$$p_t(a_1\dots a_n)=\sum_{t_1+t_2+1=t} p_t(a_1\dots a_{n-1})q_{a_{n-1}a_n}\varphi_{a_n}^{t_2}. \tag{2.3}$$

Hence,

$$p_t(a_1\dots a_n)=\sum_{t_0+t_1+\cdots t_n+n=t}\mathsf{P}\{\xi_{t_0}=*\mid \xi_0=*\}q_{*a_1}\varphi_{a_1}^{t_1}\dots q_{a_{n-1}a_n}\varphi_{a_n}^{t_n}.$$

It is easy to derive the following equations for φ_a^t

$$\begin{aligned}\varphi_a^0 &= 1,\\ \varphi_a^t &= \sum_{b,t_1+t_2+2=t} q_{ab}\varphi_b^{t_1}q_b\varphi_a^{t_2},\quad t>0.\end{aligned} \tag{2.4}$$

The lemma is proved. □

Remark. We have $\varphi_a = \sum_t \varphi_a^t = 1/q_a$. Indeed, by the recurrence of the process ξ_t we have that

$$\sum_{t\geq 0} \varphi_a^t q_a = \mathsf{P}\{\xi_t = * \text{ for some } t \mid \xi_0 = a\} = 1.$$

It is useful to use a more compact notation. Define for $t > 0$, $a \in \{*\} \cup R$, $b \in R$,

$$h_{ab}^t = q_{ab}\varphi_a^{t-1}, \quad h_{ab}(z) = \sum_{t>0} z^t h_{ab}^t.$$

From the previous remark, we have

$$h_{ab}(1) = q_{ab}/q_a.$$

By null-recurrence, the maximum eigenvalue of the matrix $H = \{h_{ab}(1)\}_{a,b\in R}$ equals 1. In terms of $h_{ab}(z)$ we get

$$p_{a_1\ldots a_n} = \varphi_* h_{*a_1} h_{a_1a_2} \ldots h_{a_{n-1}a_n}. \tag{2.5}$$

The following more general recurrent formula can be derived similarly to the proof of (2.3).

Lemma 2.2. *Let $a \in R, \beta = b_0b_1\ldots b_k$, $\alpha \in a_0a_1\ldots a_n$, $b_0 = a_0 = *$, $k, n \geq 0$ and $n = |\alpha|$. Then*

$$\begin{aligned} \mathsf{P}\{\xi_t = \alpha a \mid \beta\} &= \mathbf{1}_{\alpha a}(b_0\ldots b_{\min(n+1,k)})\mathsf{P}\{m_t = n+1\} \\ &\quad + \sum_{t_1+t_2=t} \mathsf{P}\{\xi_{t_1} = \alpha \mid \beta\} h_{a_n a}^{t_2}, \end{aligned} \tag{2.6}$$

where $m_t = \min_{s\leq t}|\xi_s|$.

In the following subsections we will prove the main theorems.

2.1. Stabilisation law

Here we will give the proof of Theorem 1.1.

To simplify formulae, let us consider the case that $\delta = a \in R$. To demonstrate the main ideas, we will start with the case $\xi_0 = *$.

Let $p_{\text{Last}}^t(a) = \mathsf{P}\{\text{Last}_0(\xi_t) = a \mid \xi_0 = *\}$. Then

$$p_{\text{Last}}^t(a) = \sum_{n\geq 0} \sum_{\substack{a_1,\ldots,a_{n-1}\in R,\\ a_0=*}} p_t(a_0a_1\ldots a_{n-1}a).$$

From (2.6) we get

$$\begin{aligned} p^t_{\mathrm{Last}}(a) &= \sum_{\substack{b\in R\cup\{*\},\\ t_1+t_2=t}} p^{t_1}_{\mathrm{Last}}(b) h^{t_2}_{ba} \\ &= \mathsf{P}\{\xi_t = a \mid \xi_0 = *\} + \sum_{\substack{b\in R,\\ t_1+t_2=t}} p^{t_1}_{\mathrm{Last}}(b) h^{t_2}_{ba}. \end{aligned} \tag{2.7}$$

The first term tends to zero by non-ergodicity of the Markov chain. Since

$$\sum_{a\in R} p^t_{\mathrm{Last}}(a) = 1 - \mathsf{P}\{\xi_t = * \mid \xi_0 = *\},$$

we similarly obtain

$$\lim_{t\to\infty} \sum_{a\in R} p^t_{\mathrm{Last}}(a) = 1.$$

Hence, for any subsequence t_k such that $p^{t_k}_{\mathrm{Last}}(a)$ tends to some limit, l^0_a say, for all $a \in R$, we have

$$\sum_{a\in R} l^0_a = 1.$$

We want to prove that $l^0_a = l_a$, where $l = \{l_a\}_{a\in R}$ is the left eigenvector of H

$$lH = l$$

with $\sum_a l_a = 1$.

Indeed, by diagonalisation we can find a subsequence $\{p^{t_n}_{\mathrm{Last}}(a)\}_{a\in R, l>0}$ such that for all $a \in R$ and $k \geq 0$ there exist constants l^{-k}_a with

$$\lim_{n\to\infty} p^{t_n-k}_{\mathrm{Last}}(a) = l^{-k}_a$$

and

$$\sum_a l^{-k}_a = 1.$$

Passing to the limit in (2.7) along the subsequence t_n, we get for all a, k that

$$l^{-k}_a = \sum_{b,t>0} l^{-k-t}_b h^t_{ba}.$$

Let $\{\epsilon^{(t)}\}_{t\geq 0}$ be a sequence with

$$h^t_{ab} \geq \epsilon^{(t)} \geq 0,$$

for all $a, b \in \{1, \ldots, r\}, t > 0$ and

$$\sum_{t>0} \epsilon^{(t)} = \epsilon > 0. \tag{2.8}$$

Then for any $k \le 0$ we have

$$\begin{aligned}
|l_a^k - l_a| &= \Big| \sum_{b,t>0} (l_b^{k-t} - l_b) h_{ba}^t \Big| \\
&= \Big| \sum_{b,t>0} (l_b^{k-t} - l_b)(h_{ba}^t - \epsilon^{(t)}) + \sum_{t>0} \epsilon^{(t)} \sum_b (l_b^{k-t} - l_b) \Big| \\
&= \Big| \sum_{b,t>0} (l_b^{k-t} - l_b)(h_{ba}^t - \epsilon^{(t)}) \Big| \\
&\le \sum_{b,t>0} |l_b^{k-t}/l_b - 1| l_b (h_{ba}^t - \epsilon^{(t)}) \\
&\le l_a (1 - r\epsilon/l_a) \sup_{t>0, b\in R} |l_b^{k-t}/l_b - 1|.
\end{aligned}$$

So for $c = \max_{a\in R}(1 - r\epsilon/l_a) < 1$ and all $k \le 0$ we have

$$|l_a^k/l_a - 1| \le c \sup_{t<k, b\in R} |l_b^t/l_b - 1|. \tag{2.9}$$

Using this inequality for each term in the right-hand side of (2.9) we get

$$|l_a^k/l_a - 1| \le c^2 \sup_{t<k-1, b\in R} |l_b^t/l_b - 1|.$$

And after n such steps we have

$$|l_a^k/l_a - 1| \le c^{n+1} \sup_{t<k-n, b\in R} |l_b^t/l_b - 1|.$$

It follows that $l_a^k = l_a$ for all k, a.

This kind of argument is typical also for the more difficult cases that we will consider later on.

Next we consider the general case. Let $\beta = b_0 b_1 \dots b_n$ be the initial state of ξ_t, i.e. $\xi_0 = \beta$. Define $p_{\mathrm{Last}}^t(a) = \mathsf{P}\{\mathrm{Last}_0(\xi_t) = a \mid \xi_0 = \beta\}$. Then from (2.6) we have

$$\begin{aligned}
p_{\mathrm{Last}}^t(a) &= \sum_{k=0}^n \mathsf{P}\{|\xi_t| = m_t = k \mid \xi_0 = \beta\} \mathbf{1}_a(b_k) + \sum_{\substack{b\in R\cup\{*\},\\ t_1+t_2=t}} p_{\mathrm{Last}}^{t_1}(b) h_{ba}^{t_2} \\
&= \sum_{k=0}^n \mathsf{P}\{|\xi_t| = m_t = k \mid \xi_0 = \beta\} \mathbf{1}_a(b_k) + \mathsf{P}\{\xi_t = a, \xi_{t-1} = * \mid \xi_0 = \beta\} \\
&\quad + \sum_{\substack{b\in R, t_1\le t/2\\ t_1+t_2=t}} p_{\mathrm{Last}}^{t_1}(b) h_{ba}^{t_2} + \sum_{\substack{b\in R, t_1> t/2\\ t_1+t_2=t}} p_{\mathrm{Last}}^{t_1}(b) h_{ba}^{t_2},
\end{aligned}$$

where $m_t = \min_{s\le t} |\xi_s|$. Define

$$\begin{aligned} A_t(\beta,a) &= \sum_{k=0}^{n} \mathsf{P}\{|\xi_t| = m_t = k \mid \xi_0 = \beta\}\mathbf{1}_a(b_k) \\ &\quad + \mathsf{P}\{\xi_t = a, \xi_{t-1} = * \mid \xi_0 = \beta\} + \sum_{\substack{b\in R, t_1\le t/2 \\ t_1+t_2=t}} p^{t_1}_{\text{Last}}(b) h^{t_2}_{ba}. \end{aligned}$$

The next inequality is derived similarly to (2.9)

$$\begin{aligned} \left|\frac{p^t_{\text{Last}}(a)}{l_a} - 1\right| &\le \Big(1 - \frac{r}{l_a}\sum_{s=1}^{t/2} \varepsilon^{(s)}\Big) \max_{\substack{b\in R, \\ t/2\le s<t}} \left|\frac{p^s_{\text{Last}}(b)}{l_b} - 1\right| \\ &\quad + \frac{1}{l_a} A_t(\beta,a) + \frac{1}{l_a} \sum_{b\in R, s\ge t/2} l_b h^s_{ba}. \qquad (2.10) \end{aligned}$$

Remark that $\varepsilon^{(1)}$ can be chosen positive, since $h^1_{ba} = q_{ba} > 0$ for all $b, a \in R$. Hence,

$$(1 - \frac{r}{l_a}\sum_{s=1}^{t/2} \varepsilon^{(s)}) < c, \text{ for all } a \in R,$$

for some $c < 1$. Suppose that there exists a non increasing function B_t with $B_t \to 0$, as $t \to \infty$, and for all $\beta \in \mathcal{A}, a \in R$

$$\frac{1}{l_a} A_t(\beta,a) + \frac{1}{l_a} \sum_{b\in R, s\ge t/2} l_b h^s_{ba} < B_t.$$

Define

$$C_t = \sup_{\beta} \max_{a\in R} \left|\frac{\mathsf{P}\{\text{Last}_0(\xi_t) = a \mid \xi_0 = \beta\}}{l_a} - 1\right|.$$

From (2.10) we get

$$C_t \le B_t + c \max_{t/2\le s<t} C_s.$$

Hence

$$\begin{aligned} \limsup_{t\to\infty} C_t &\le c \limsup_{t\to\infty} \max_{t/2\le s<t} C_s \\ &\le c \lim_{t\to\infty} \sup_{s\ge t/2} C_s \\ &= c \limsup_{t\to\infty} C_t. \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} C_t = 0.$$

We should prove therefore that such B_t exists. For some terms this follows from convergence of $\sum_{t\geq 0} h_{ba}^t$ and the inequality

$$\sum_{\substack{b\in R, t_1\leq t/2\\ t_1+t_2=t}} p_{\mathrm{Last}}^{t_1}(b) h_{ba}^{t_2} + \frac{1}{l_a} \sum_{b\in R, s\geq t/2} l_b h_{ba}^s < C \sum_{b\in R, s\geq t/2} h_{ba}^s, \quad \text{for some } C > 0.$$

For other terms this follows from the following lemma.

Lemma 2.3. *We have*

$$\mathsf{P}\{|\xi_t| = m_t \mid \xi_0 = \beta\} \to 0, \text{ as } t \to \infty \text{ uniformly in } \beta \in \mathcal{A}.$$

Proof. Let $\beta = b_0 b_1 b_2 \ldots b_n$, $b_0 = *$. Let further $\tau_1, \tau_2, \ldots, \tau_n$ be the random moments defined by

$$\tau_k = \min\{t : \xi_t = b_0 b_1 b_2 \ldots b_{k-1} \mid \xi_0 = b_0 b_1 b_2 \ldots b_{k-1} b_k\}.$$

In other words, τ_k is the time that symbol b_k is deleted. It is clear that $\tau_1, \tau_2, \ldots, \tau_n$ are independent and have distribution

$$\begin{aligned} \mathsf{P}\{\tau_k = t\} &= q_{b_k} \varphi_{b_k}^{t-1}, \quad \text{for } t \geq 1, \\ \mathsf{P}\{\tau_k = 0\} &= 0. \end{aligned}$$

Note also, that

$$\mathsf{P}\{\tau_k = 2t\} = 0, \text{ for } t \geq 0. \tag{2.11}$$

Let σ be the random time defined by

$$\sigma = \min\{t > 0 : \xi_t = * \mid \xi_0 = *\}.$$

So σ is the first time (after 0) of hitting $*$, when starting at $*$. Let $\sigma_1, \sigma_2, \ldots$ be a sequence of identically distributed random moments with the same distribution as σ. Then

$$\begin{aligned} \mathsf{P}\{\sigma_k = 0\} &= 0, \\ \mathsf{P}\{\sigma_k = 1\} &= 0, \\ \mathsf{P}\{\sigma_k = t\} &= \sum_{a\in R} q_{*a} q_a \varphi_a^{t-2}, \quad \text{for } t \geq 2. \end{aligned}$$

Remark that

$$\mathsf{P}\{\sigma_k = 2t+1\} = 0, \text{ for } t \geq 0. \tag{2.12}$$

We can write now

$$\begin{aligned} \mathsf{P}\{|\xi_t| = m_t \mid \xi_0 = \beta\} &= \mathsf{P}\{|\xi_t| = m_t, |\beta| > m_t > 0 \mid \xi_0 = \beta\} \\ &\quad + \mathsf{P}\{|\xi_t| = m_t = 0 \mid \xi_0 = \beta\}. \end{aligned} \tag{2.13}$$

Let us show that the first term in the right-hand side tends to 0, uniformly in β.

$$\begin{aligned}
&\mathsf{P}\{|\xi_t| = m_t > 0 \mid \xi_0 = \beta\} \\
&= \sum_{k=2}^{n} \sum_{t_1+t_2=t} \mathsf{P}\{\tau_n + \cdots + \tau_k = t_1\}\varphi_{b_{k-1}}^{t_2} \\
&= \sum_{k=2}^{n} \frac{1}{q_{b_{k-1}}} \sum_{t_1+t_2=t} \mathsf{P}\{\tau_n + \cdots + \tau_k + \tau_{k-1} = t+1\} \\
&\leq \max_{b\in R}\{1/q_b\}\mathsf{P}\{\tau_n + \cdots + \tau_k = t+1, \text{ for some } k \geq 1\}. \quad (2.14)
\end{aligned}$$

Corollary 3.1 implies that the distributions of $\tau_n, \ldots, \tau_1$ satisfy the conditions of Theorem 4.1 in the Appendix. This theorem immediately implies that (2.14) converges uniformly in β.

The second term in (2.13) can be bounded similarly. Rewrite

$$\begin{aligned}
&\mathsf{P}\{|\xi_t| = m_t = 0 \mid \xi_0 = \beta\} \\
&= \mathsf{P}\{\text{there exists } k \geq 0 : \tau_n + \cdots + \tau_1 + \sigma_1 + \cdots + \sigma_k = 1\}. \quad (2.15)
\end{aligned}$$

The conditions of Theorem 4.1 do not hold because of (2.11) and (2.12). Therefore, we can not use this theorem directly and we should rewrite (2.15) in the following way

$$\begin{aligned}
&\mathsf{P}\{|\xi_t| = m_t = 0 \mid \xi_0 = \beta\} \\
&= \mathsf{P}\{(\tau_n + 1) + \cdots + (\tau_1 + 1) + \sigma_1 + \cdots + \sigma_k = 1 + n, \text{ for some } k \geq 0\}.
\end{aligned}$$

Again Corollary 3.1 implies that the random variables $\tau_n+1, \ldots, \tau_1+1, \sigma_1, \sigma_2, \ldots$ satisfy the conditions of Theorem 4.1. Hence, there exist functions ψ_t with $\lim_{t\to\infty} \psi_t = 0$ and for any β

$$\mathsf{P}\{|\xi_t| = m_t = 0 \mid \xi_0 = \beta\} \leq \psi_{t+|\beta|} = \sup_{s\geq t} \psi_s = \tilde{\psi}_t.$$

Obviously $\lim_{t\to\infty} \tilde{\psi}_t = 0$. This proves Lemma 2.3 and hence also Theorem 1.1. □

2.2. Mixing property

Next we prove Theorem 1.2.

By definition

$$\begin{aligned}
\pi_{t,n}(a_1 \ldots a_n) &= \frac{1}{Z_{t,n}} p_t(a_1 \ldots a_n) \\
&= \frac{1}{Z_{t,n}} \sum_{t_0+\cdots+t_n=t} \varphi_*^{t_0} h_{*a_1}^{t_1} h_{a_1a_2}^{t_2} \ldots h_{a_{n-1}a_n}^{t_n} \\
&= \frac{1}{Z_{t,n}} \sum_{t_0+\cdots+t_n=t} Z_{t_0,t_1,\ldots,t_n} \frac{1}{Z_{t_0,t_1,\ldots,t_n}} \varphi_*^{t_0} h_{*a_1}^{t_1} h_{a_1a_2}^{t_2} \ldots h_{a_{n-1}a_n}^{t_n},
\end{aligned}$$

where
$$Z_{t_0 t_1 \dots t_n} = \sum_{a_0,\dots,a_n} \varphi_*^{t_0} h_{*a_1}^{t_1} h_{a_1 a_2}^{t_2} \dots h_{a_{n-1}a_n}^{t_n}.$$

So it is sufficient to prove that the correlations (defined similarly to Theorem 1.2) of the measures

$$\pi_{t_0,t_1,\dots,t_n}(a_1 \dots a_n) = \frac{1}{Z_{t_0,t_1,\dots,t_n}} \varphi_*^{t_0} h_{*a_1}^{t_1} h_{a_1 a_2}^{t_2} \dots h_{a_{n-1}a_n}^{t_n},$$

decay exponentially quickly, uniformly in $t_0, t_1, \dots, t_n$. Let $\{\sigma_1, \dots, \sigma_n\}$ be random variables with values in R and with distribution $\pi_{t_0,t_1,\dots,t_n}$. This sequence can be interpreted as an inhomogeneous Markov chain on the state space R evolving till time n. Its transition probabilities at time $k \geq 1$ can be easily calculated: for $a, b \in R$, $k \geq 1$, we have

$$\mathsf{P}\{\sigma_k = b \mid \sigma_{k-1} = a\} = \frac{h_{ab}^{t_k} \sum\limits_{a_{k+1},\dots,a_n} h_{ba_{k+1}}^{t_{k+1}} \dots h_{a_{n-1}a_n}^{t_n}}{\sum\limits_{a_k} h_{aa_k}^{t_k} \sum\limits_{a_{k+1},\dots,a_n} h_{a_k a_{k+1}}^{t_{k+1}} \dots h_{a_{n-1}a_n}^{t_n}}. \tag{2.16}$$

In Section 3 we will derive the existence of $\varepsilon > 0$, such that for all $a, b, c, d \in R$, $k = 1, \dots, n$,
$$h_{ab}^{t_k} / h_{cd}^{t_k} > \varepsilon. \tag{2.17}$$

Then (2.17) and (2.16) imply the following estimates for these transition probabilities: for any $k \geq 1, a, b \in R$,

$$\mathsf{P}\{\sigma_k = b \mid \sigma_{k-1} = a\} > \varepsilon^2 / r. \tag{2.18}$$

So this chain has the exponential mixing property as defined in the statement of Theorem 1.2.

Recall the notation of this theorem. Then by virtue of (2.18) there exist $C_1 > 0, 0 < C_2 < 1$, such that for any $0 \leq k < m \leq n$ and any events $A \in \mathcal{F}_k$, $B \in \mathcal{F}^m$
$$\left|\mathsf{P}(A \mid B) - P(A)\right| < C_1 C_2^{m-k}.$$

The coefficients C_1, C_2 depend on ε but do not depend on $t_0, t_1, \dots, t_n$. □

2.3. Law of Large Numbers

This subsection will prove Theorem 1.3.

First we will prove convergence of the expectation of $n_a(t)/n(t)$ to some limit. Then we will show that the variance of $n_a(t)/n(t)$ tends to 0. We use the same method as in the proof of Theorem 1.1: we will derive some equations for the limit points of the sequence $\mathsf{E}(n_a(t)/n(t))$, and then we will check that these equations have a unique solution.

Lemma 2.4.

$$\lim_{t\to\infty} \mathsf{E}\frac{n_a(t)}{n(t)} = l_a r_b,$$

where $l = \{l_a\}_{a\in R}$, $r = \{r_a\}_{a\in R}$ *are left and right eigenvectors of the matrix* $H = \{h_{ab}(1)\}_{a,b\in R}$

$$lH = l,$$
$$Hr = r,$$

with $\sum_a l_a r_a = 1$.

Proof. Let us write

$$n_a(t) = \sum_{k=1}^{n(t)} \mathbf{1}\{\xi_t(k) = a\}.$$

Define

$$n_{ab}(t) = \sum_{k=1}^{n(t)-1} \mathbf{1}\{\xi_t(k) = a, \xi_t(k+1) = b\},$$

then

$$\frac{n_a(t)}{n(t)} = \sum_b \frac{n_{ab}(t)}{n(t)-1} + O(\frac{1}{n(t)}).$$

It is therefore sufficient to prove that

$$\lim_{t\to\infty} \mathsf{E}\frac{n_{ab}(t)}{n(t)-1} = l_a r_b.$$

From the definition of $n_{ab}(t)$ we have

$$\begin{aligned}\mathsf{E}\frac{n_{ab}(t)}{n(t)} &= \sum_{n=2}^{\infty} \frac{1}{n-1} \sum_{k=1}^{n-1} \mathsf{P}\{\xi_t(k) = a, \xi_t(k+1) = b, |\xi_t| = n\} \\ &= \sum_{n=2}^{\infty} \frac{1}{n-1} \sum_{k=1}^{n-1} \sum_{\substack{a_1,\dots,a_n \\ t_0+\dots+t_n=t}} \varphi_*^{t_0} h_{*a_1}^{t_1} h_{a_1a_2}^{t_2} \dots h_{a_{k-1}a}^{t_k} h_{ab}^{t_{k+1}} h_{ba_{k+2}}^{t_{k+2}} \dots h_{a_{n-1}a_n}^{t_n}.\end{aligned}$$

Define

$$g_{ab}(t) = \sum_{n=2}^{\infty} \frac{1}{n-1} \sum_{k=1}^{n-1} \sum_{\substack{a_1,\dots,a_n \\ t_0+\dots+t_n=t}} \varphi_*^{t_0} h_{*a_1}^{t_1} h_{a_1a_2}^{t_2} \dots h_{a_{k-1}a}^{t_k} h_{ba_{k+2}}^{t_{k+2}} \dots h_{a_{n-1}a_n}^{t_n}, \tag{2.19}$$

and so

$$\mathsf{E}\frac{n_{ab}(t)}{n(t)-1} = \sum_{t_1+t_2=t} g_{ab}(t_1) h_{ab}^{t_2}.$$

We want to prove the existence of the following limit

$$\lim_{t\to\infty} g_{ab}(t) = l_a r_b.$$

To this end, we use the same idea as in proof of Theorem 1.1. This means that we will construct a sequence of limit points satisfying some linear equations. We will prove subsequently that the solution of these equations is unique.

Remark that

$$\sum_{ab}\sum_{t_1+t_2=t} g_{ab}(t_1)h_{ab}^{t_2} = \sum_{ab} \mathsf{E}\frac{n_{ab}(t)}{n(t)-1} = 1.$$

Let t_k be a subsequence, such that

$$\begin{array}{rcl} g_{ab}(t_k) & \to & g_{ab}^{0}, \\ g_{ab}(t_{k-1}) & \to & g_{ab}^{-1}, \\ & \cdots & \\ g_{ab}(t_{k-n}) & \to & g_{ab}^{-n}, \quad \text{for } n>0, \end{array}$$

as $k\to\infty$. From the last remark, we have for any $t\le 0$ that

$$\sum_{ab}\sum_{t_1+t_2=t} g_{ab}^{t_1}h_{ab}^{t_2} = 1. \tag{2.20}$$

But by the definition of $g_{ab}(t)$ we have for any $t<0$

$$g_{ab}^{t} = \sum_{\substack{c,d,\\ t_1+t_2+t_3=t}} g_{cd}^{t_1}h_{ca}^{t_2}h_{bd}^{t_3}.$$

Using

$$h_{cabd}^{t} = \sum_{t_1+t_2=t} h_{ca}^{t_1}h_{bd}^{t_2},$$

we can rewrite this as

$$g_{ab}^{t} = \sum_{\substack{c,d,\\ t_1+t_2=t}} g_{cd}^{t_1}h_{cabd}^{t_2}.$$

Let us now prove that $g_{ab}^{t} = l_a r_b$. To this end, write

$$\left|g_{ab}^{t} - l_a r_b\right| = \left|\sum_{\substack{c,d,\\ t_1+t_2=t}} (g_{cd}^{t_1} - l_c r_d)h_{cabd}^{t_2}\right|.$$

Using (2.20) we get

$$\sum_{t_1+t_2=t} g_{cd}^{t_1}h_{cd}^{t_2} = \sum_{t_1+t_2=t} l_c r_d h_{cd}^{t_2} = 1,$$

and subsequently using Corollary 3.1, we can show the existence of $\varepsilon > 0$, such that for all $a, b, c, d \in R, t \geq 0$,

$$h^t_{cabd} > \varepsilon h^t_{cd}.$$

Therefore,

$$\begin{aligned}
\Big|\sum_{\substack{c,d,\\ t_1+t_2=t}} (g^{t_1}_{cd} - l_c r_d) h^{t_2}_{cabd}\Big| &= \Big|\sum_{\substack{c,d,\\ t_1+t_2=t}} (g^{t_1}_{cd} - l_c r_d)(h^{t_2}_{cabd} - \varepsilon h^{t_2}_{cd})\Big| \\
&\leq \sum_{\substack{c,d,\\ t_1+t_2=t}} \Big|\frac{g^{t_1}_{cd}}{l_c r_d} - 1\Big| l_c r_d (h^{t_2}_{cabd} - \varepsilon h^{t_2}_{cd}) \\
&\leq (l_a l_b - \varepsilon) \max_{\substack{c,d,\\ t_1<t}} \Big|\frac{g^{t_1}_{cd}}{l_c r_d} - 1\Big|.
\end{aligned}$$

In other words, we get for some constant $c < 1$ and any $t \leq 0$ that

$$\Big|\frac{g^t_{ab}}{l_a r_b} - 1\Big| \leq c \max_{\substack{c,d,\\ t_1<t}} \Big|\frac{g^{t_1}_{cd}}{l_c r_d} - 1\Big|.$$

Hence, $g^t_{ab} = l_a r_b$, for all $t \leq 0$. □

Lemma 2.5.

$$\lim_{t\to\infty} \mathrm{D}\frac{n_a(t)}{n(t)} = 0.$$

Proof. We have to prove that

$$\lim_{t\to\infty} \mathsf{E}\Big(\frac{n_a(t)}{n(t)}\Big)^2 = (l_a r_b)^2.$$

With some small modifications this can be derived in the same way as calculating the expectation of $n_a(t)/n(t)$. First write

$$\Big(\frac{n_a(t)}{n(t)}\Big)^2 = \frac{1}{(n(t)-1)^2} \sum_{b,b_1} n_{ab}(t) n_{ab_1}(t) + O\Big(\frac{1}{n(t)}\Big).$$

It is convenient to consider a more general case and prove that

$$\mathsf{E}\frac{n_{ab}(t) n_{a_1 b_1}(t)}{(n(t)-1)^2} \to l_a r_b l_{a_1} r_{b_1} \quad \text{as } t \to \infty.$$

From

$$\begin{aligned}
&n_{ab}(t) n_{a_1 b_1}(t) \\
&\quad = \sum_{k=1}^{n(t)-1} \sum_{m=1}^{n(t)-1} \mathbf{1}\{\xi_t(k) = a, \xi_t(k+1) = b\} \mathbf{1}\{\xi_t(m) = a_1, \xi_t(m+1) = b_1\},
\end{aligned}$$

we get

$$\mathsf{E}\frac{n_{ab}(t)n_{a_1b_1}(t)}{(n(t)-1)^2} = \sum_{t_1+t_2+t_3=t} g_{aba_1b_1}(t_1)h_{ab}^{t_2}h_{a_1b_1}^{t_3},$$

where $g_{aba_1b_1}$ is defined similarly to (2.19). Now we can construct a sequence of limit points $g_{aba_1b_1}^t$, $t \le 0$, satisfying the equations

$$g_{aba_1b_1}^t = \sum_{\substack{c,d,c_1,d_1 \\ t_1+t_2+t_3+t_4+t_5=t}} g_{cdc_1d_1}^{t_1} h_{ca}^{t_2} h_{bd}^{t_3} h_{ca_1}^{t_4} h_{b_1d}^{t_5}.$$

In the same way as we did in the above, it is easy to prove that these equations have a unique solution, which is given by

$$g_{aba_1b_1}^t = l_a r_b l_{a_1} r_{b_1}.$$

□

2.4. Central Limit Theorem

Next we prove Theorem 1.4.

The main idea is to prove central limit theorem first for some linear combination of $n_a(t)$, where $n_a(t)$ is the number of symbols a at time t. This can be achieved using a general form of the central limit theorem for martingales. All such general theorems have rather restrictive conditions and in order to check them, we shall essentially use Theorem 1.1. After this, we can easily derive Theorem 1.4 from Theorem 1.3.

In a null recurrent case (which we consider), there is a positive vector $\{e_a\}_{a\in R}$ (see (1.2)), such that for the function

$$f(\xi_t) = \sum_{a=1}^{r} e_a n_a(t)$$

the following identity holds

$$\mathsf{E}(f(\xi_{t+1}) \mid \xi_t) = f(\xi_t), \text{ if } \xi_t \ne \emptyset.$$

So $f(\xi_t)$ is a martingale “up to” jumps from the empty string. To obtain a martingale, we can do a symmetrisation by, for example, assigning the sign “$-$” or “$+$” with equal probabilities to the string, each time that ξ_t jumps from the empty string. Let ν_t be the number of times that $\xi_k = \emptyset$ before time t, i.e.

$$\nu_t = \#\{k : \xi_k = \emptyset, k \le t\}.$$

Let $\{\sigma_k\}$ be a sequence of independent random variables with values in $\{-1, 1\}$, such that

$$\sigma_k = 1 \text{ with probability } 1/2.$$

Then the sequence of random variables

$$\eta_t = \sigma_{\nu_t} \sum_{a=1}^{r} e_a n_a(t)$$

is a martingale. We shall use now the central limit theorem for martingales (see [10] Chapter 7, Section 8, Theorem 1). The first two conditions of this theorem follow from the boundedness of jumps for the function $f(\xi_t)$. The third follows from the stabilisation law (Theorem 1.1) and from the law of large numbers for weakly dependent random variables (see [9]). Let us consider this condition in detail. We should check (see [10] Chapter 7, Section 8, Theorem 1, condition C) that for any $0 < x \leq 1$

$$\sum_{k=1}^{[tx]} \mathrm{D}\Big[\frac{\eta_k - \eta_{k-1}}{\sqrt{t}} \;\Big|\; \mathcal{F}_{k-1}\Big] \xrightarrow{\mathsf{P}} c_x^2, \text{ as } t \to \infty, \tag{2.21}$$

where $c_x^2 \geq 0$ and $\mathcal{F}_k$ is the σ-algebra generated by $\{\xi_0, \ldots, \xi_k\}$. The random variable $\mathrm{D}\,[\eta_k - \eta_{k-1} \mid \mathcal{F}_{k-1}]$ is a positive function of the last symbol of the string:

$$\mathrm{D}\,[\eta_k - \eta_{k-1} \mid \mathcal{F}_{k-1}] = F(L(\xi_{k-1})),$$

where $L(\xi_{k-1}) = a_n$, if $\xi_{k-1} = a_1 \ldots a_n$. So condition (2.21) can be written as

$$\frac{1}{t} \sum_{k=1}^{t} f(L(\xi_{k-1})) \xrightarrow{\mathsf{P}} c^2, \text{ as } t \to \infty.$$

For this, it is sufficient to prove that for any $a \in R$

$$\frac{1}{t} \sum_{k=1}^{t} 1_a(L(\xi_k)) \xrightarrow{\mathsf{P}} c_a, \text{ as } t \to \infty. \tag{2.22}$$

This is just the law of large numbers applied to the sequence of random variables $\{\zeta_k = 1_a(L(\xi_k))\}_{k \geq 1}$. Hence we should prove that this sequence indeed obeys the law of large numbers.

By virtue of Theorem 1.1, the sequence ζ_k has the $*$-mixing property (here we follow the terminology of [9]). Denoting by $\mathcal{F}_{[k,n]}$ the σ-algebra generated by $\{\zeta_k, \ldots, \zeta_n\}$, this means that there exists a non-increasing function $\psi_t \to 0$ as $t \to \infty$, such that for any $A \in \mathcal{F}_{[1,n]}$, $B \in \mathcal{F}_{[n+t,n+t]}$ and any n, t

$$|\mathsf{P}(B) - \mathsf{P}(B)| < \psi_t \mathsf{P}(A)\mathsf{P}(B). \tag{2.23}$$

But Theorem 8.2.1 of [9] states that the law of large numbers holds for a sequence with the $*$-mixing property under some additional moment conditions (which are trivial in our case due to $\zeta_k \leq 1$).

As a consequence we obtain (2.22) and so condition (2.21) holds.

Thus, $\lim_{t\to\infty} \eta_t/\sqrt{t}$ has the normal distribution. Write

$$\frac{n_t}{\sqrt{t}} = \frac{n_t}{|\eta_t|}\frac{|\eta_t|}{\sqrt{t}} = \frac{n_t}{\sum\limits_{a=1}^{r} e_a n_a(t)}\frac{|\eta_t|}{\sqrt{t}}.$$

By our law of large numbers Theorem 1.3

$$\frac{n_t}{\sum\limits_{a=1}^{r} e_a n_a(t)} \to \text{const},$$

and so our central limit theorem follows, since

$$\lim_{t\to\infty}\frac{n_t}{\sqrt{t}} = \text{const} \lim_{t\to\infty}\frac{|\eta_t|}{\sqrt{t}}.$$

□

3. Generating functions

In this section we consider some analyticity properties of the functions $\varphi_a(z)$ and we will also give another proof of Theorem 1.4 using complex variable techniques.

By their definition, the functions $\varphi_a(z)$ are analytical on $\{z : |z| < 1\}$. Equations (2.1) therefore imply the following expansion for $\varphi_a(z)$

$$\varphi_a(z) = \sum_{s\geq 0}\varphi_a^s z^s.$$

All odd coefficients φ_a^{2s+1} are equal to 0.

It is simpler to work with the functions $\{u_a(z)\}_{a\in R}$ defined by

$$u_a(z) = q_a \sum_{s\geq 0}\varphi_a^{2s} z^s.$$

Let $\tau_a = \min\{t : \xi_t = * \mid \xi_0 = a\}$. Then $u_a(z^2) = \mathsf{E} z^{\tau_a - 1}$. The generating functions $u_a(z)$ satisfy the equations

$$u_a(z) = q_a + z\sum_b q_{ab} u_b(z) u_a(z). \tag{3.1}$$

Introduce the following notation. For a given sequence $\{x_a\}_{a\in R}$ we denote by $\vec{x}$ the vector with components x_a and by $\widehat{X}$ the diagonal matrix with diagonal elements x_a. By I we denote the unit matrix and by $\vec{1}$ the vector with all components equal to 1.

Equation (3.1) can thus be rewritten as

$$\vec{u}(z) = \vec{q} + z\widehat{U}(z)Q\vec{u}(z).$$

We know $\vec{u}(1) = \vec{1}$ and $\vec{u}(z)$ to be analytic on $\{z : |z| < 1\}$ and continuous on $\{z : |z| \leq 1\}$. But $\vec{u}(1)$ has a singularity at point $\vec{1}$, since from null recurrence we have

$$u_a'(1) = \mathsf{E}(\tau_a - 1)/2 = \infty.$$

Since the $u_a(z)$ satisfy a system of algebraic equations, these functions have only algebraic singularities. Hence, it follows that the singularity at 1 is an algebraic singularity.

Let us recall one useful theorem (a special case of the Darboux theorem, see [7], [8]) and some related necessary definitions.

Suppose that the function $f(z)$ has a singularity at z_0. This singularity is called algebraic if $f(z)$ can be written as a function that is analytic near z_0, plus a finite sum of terms of the form

$$(z - z_0)^{-\omega} g(z), \tag{3.2}$$

where g is a function that is analytic and non-zero near z_0 and ω is a complex number not equal $0, -1, -2, \ldots$ Call the real part of ω the weight of the term (3.2).

Theorem 3.1. *Suppose that $A(z) = \sum_{n\geq 0} a_n z^n$ is analytic near 0 and has only algebraic singularities on its circle of convergence. Let w be maximum of the weights at these singularities. Denote by z_k, ω_k and g_k the values of z_0, ω and g for the terms of the form (3.2) having weight w. Then*

$$a_n - \frac{1}{n} \sum_k \frac{g_k(z_k)}{\Gamma(\omega_k) z_k^n} = o(r^{-r} n^{w-1}),$$

where $r = |z_k|$ is the radius of convergence of $A(z)$, and $\Gamma(s)$ is the gamma-function.

As we mentioned in the above, the functions $u_a(z)$ have only algebraic singularities, because they solve a system of algebraic equations. The latter also implies that the ωs in the terms of the form (3.2) are rational.

Lemma 3.1. *For each $a \in R$, the point 1 is the only singular point of the function $u_a(z)$, $a \in R$, on the unit circle.*

Proof. Let z_0, $|z_0| = 1$, be a singular point of one of the functions $u_a(z)$. For z, $|z| \leq 1$, we have

$$1 = \frac{q_a}{u_a(z)} + z \sum_b q_{ab} u_b(z)$$

or, in vector form,

$$\vec{1} = \vec{F}(\vec{u}(z)).$$

By taking the derivative in the above equality, we get

$$\sum_b q_{ab} u_b(z) = \frac{q_a}{u_a^2(z)} u_a'(z) - z \sum_b q_{ab} u_b'(z).$$

If z_0 is a singular point, then $\det \left(\frac{d}{d\vec{u}} \vec{F} \mid_{\vec{u}(z_0)} \right) = 0$. Hence the matrix

$$\widehat{Q}\widehat{U}^{-2}(z_0) - z_0 Q$$

is not invertible. This means that there exists a vector $\vec{v} = \{v_a\}_{a\in R}$, such that

$$\frac{q_a}{u_a^2(z_0)} v_a = z_0 \sum_b q_{ab} v_b.$$

Taking into account that $|u_a(z_0)| \le 1, |z_0| = 1$, we get

$$|v_a| \le |u_a(z_0)| \sum_b \frac{q_{ab}}{q_a} |v_b| \text{ for all } a \in R.$$

The matrix $\left\{ \frac{q_{ab}}{q_a} \right\}_{a,b\in R}$ is positive with maximal eigenvalue 1 (see (1.3)). Hence

$$|u_a(z_0)| \ge 1,$$

for some $a \in R$. This can only be at the point $z_0 = 1$, since $u_a(z_0)$ is a generating function. □

We will prove that the functions $u_a(z)$ have weight $(-1/2)$ at point 1. The proof of this fact is not straightforward, but a weaker assertion can be proved easily.

Lemma 3.2. *All functions $u_a(z), a \in R$, have the same weight at the point* 1.

Proof. Let the weight of the function $u_a(z)$ be w_a. Then, near 1, $u_a(z)$ can be written as

$$u_a(z) = 1 + (1-z)^{-w_a} g_a(z) + o((1-z)^{-w_a}),$$

where $g_a(z)$ is analytical near 1 and $g_a(1) \neq 0$. Moreover,

$$g_a(1) < 0,$$

for all $a \in R$, because the functions $u_a(z)$ are monotone increasing on $\{z \in \mathbf{R}, z < 1\}$. From the equations (3.1) we get

$$g_a(z) = \sum_b q_{ab} g_b(z)(1-z)^{-(w_b - w_a)} + \sum_b q_{ab} g_a(z) + o(1).$$

The functions $g_b(z)$ have the same sign near 1 ($z \in \mathbf{R}$), so they cannot be reduced one to another. On the other hand, the right side should be finite at the point 1. Hence $w_b \leq w_a$, for any $a, b \in R$, so that $w_b = w_a$. □

Theorem 3.1 and Lemmas 3.1 and 3.2 have the following corollary.

Corollary 3.1. *There exists* $\varepsilon > 0$, *such that for any* $a, b \in R$ *and* $s \geq 0$

$$\varphi_{as} > \varepsilon \varphi_{bs}.$$

Remark. It would be interesting to prove this corollary directly from equations (2.4). In this case we prove all main theorems without using analyticity properties of $u_a(z)$.

3.1. Analyticity properties

Here we prove some analyticity properties of the generating functions and give another proof of Theorem 1.4.

First of all, we need more information on the singularity at the point 1.

Theorem 3.2. *In some neighbourhood of* 1, *the functions* $u_a(z)$ *can be written as*

$$u_a(z) = \sum_{s \geq 0} u_{as}(1-z)^{s/2} = \tilde{u}_a(\sqrt{1-z}),$$

where $\tilde{u}_a(z)$ *is analytic near* 0 *and* $u_{a1} \neq 0$.

Proof. Near 1, the functions $u_a(z)$ can be written in the form

$$u_a(z) = \sum_{s \geq 0} u_{as}(1-z)^{s/n},$$

for some $n > 1$.

Let $t = (1-z)^{1/n}$. Then we get the following equations for $u_a(t)$

$$\vec{u}(t) = \vec{q} + (1 - t^n)\widehat{U}(t)Q\vec{u}(t),$$

or

$$\Big(I - (1-t^n)\sum_{s \geq 0} t^s \widehat{U}_s Q\Big) \sum_{s \geq 0} t^s \vec{u}_s = \vec{q}.$$

Also we get equations for u_{as}, namely

$$u_{a0} = u_a(z)\mid_{z=1} = 1,$$

and

$$\begin{aligned} \vec{u}_s - \sum_{l=0}^{s} \widehat{U}_l Q \vec{u}_{s-l} &= \vec{0}, \quad 1 \leq s < n, \\ \vec{u}_s - \sum_{l=0}^{s} \widehat{U}_l Q \vec{u}_{s-l} + \sum_{l=0}^{s-n} \widehat{U}_l Q \vec{u}_{s-l-n} &= \vec{0}, \quad s \geq n. \end{aligned}$$

Rewrite these equations as follows

$$\widehat{Q}\vec{u}_s - Q\vec{u}_s = \sum_{l=1}^{s-1} \widehat{U}_l Q\vec{u}_{s-l}, \quad 1 \leq s < n, \tag{3.3}$$

$$\widehat{Q}\vec{u}_s - Q\vec{u}_s = \sum_{l=1}^{s-1} \widehat{U}_l Q\vec{u}_{s-l} + \sum_{l=0}^{s-n} \widehat{U}_l Q\vec{u}_{s-l-n}, \quad s \geq n. \tag{3.4}$$

Lemma 3.3. *Let* k=min$\{s > 0 : \vec{u}_s \neq \vec{0}\}$. *Then* $n = 2k$ *and* $\vec{u}_s = \vec{0}$, *if* s *is not an integer multiple of* k.

Proof. From (3.3) we have

$$\widehat{Q}\vec{u}_k - Q\vec{u}_k = 0.$$

Hence $\vec{u}_k = u_k\vec{c}$, where $\vec{c}$ is the right eigenvector of the positive matrix $\widehat{Q}^{-1}Q$

$$\widehat{Q}^{-1}Q\vec{c} = \vec{c}.$$

By Perron – Frobenius' theorem, $\vec{c}$ is positive. Let us choose $\vec{c}$, such that $\langle\vec{c}, \vec{1}\rangle = 1$ (only for uniqueness). If $n > 2k$, we put $s = 2k$ in (3.3). We get

$$\widehat{Q}\vec{u}_{2k} - Q\vec{u}_{2k} = \widehat{U}_k Q\vec{u}_k.$$

This equation has no solutions because

$$\widehat{U}_k Q\vec{u}_k = u_k^2 \widehat{C} Q\vec{c} > \vec{0}.$$

Indeed, for some positive $\pi > 0$ we have $\pi(\widehat{Q} - Q) = 0$, see (1.3). We choose π such that $\pi\vec{1} = \vec{1}$. Hence the equation

$$(\widehat{Q} - Q)\vec{x} = \vec{y} \tag{3.5}$$

is solvable only if $\vec{y} \perp \pi$. But $\pi Q\vec{1} > 0$. Therefore, $\vec{u}'(1)$ does not exist. We will often refer to equation (3.5). Let us remark that $(\widehat{Q} - Q)^{-1}$ is well defined on the subspace $\pi^{\perp} = \{\vec{y} : \vec{y} \perp \pi\}$.

In the same way, if $n < 2k$, then we put $s = n$ in (3.3). We obtain

$$\widehat{Q}\vec{u}_n - Q\vec{u}_n = IQ\vec{1} > \vec{0}.$$

Next let us show that $\vec{u}_s \neq 0$ if and only if $s = km$. The only non-trivial case is for $k > 1$. Let $k \leq s < 2k$. Then (3.3) implies that $\vec{u}_s = u_s\vec{c}$, for some $u_s \in \mathbf{R}$. Letting $n < s \leq 2n$, then

$$\sum_{l=0}^{s-n} \widehat{U}_l Q\vec{u}_{s-l-n} = \vec{0}.$$

Hence (3.4) is solvable if

$$\pi \sum_{l=1}^{s-1} \widehat{U}_l Q \vec{u}_{s-l} = \pi \sum_{l=k}^{s-k} \widehat{U}_l Q \vec{u}_{s-l} = 0.$$

For $s = 2k+1$ we get

$$u_{k+1}(u_k \pi \widehat{C} Q \vec{c} + u_k \pi \widehat{C} Q \vec{c}) = 0 \Rightarrow u_{k+1} = 0.$$

In the same way, the equations for $s = 2k+1, 2k+2, \ldots, 2k+k-1$, yield that $u_s = 0$ for $k < s < 2k$.

Let $\vec{u}_s \neq \vec{0}$ for $s \leq mk$ only, if $s = lk, l = 1, \ldots, m$. We will show that $\vec{u}_s = \vec{0}$ if $m+k < s < (m+1)k$. From (3.4), it follows that for such s

$$\widehat{Q}\vec{u}_s - Q\vec{u}_s = \vec{0}.$$

Hence, $\vec{u}_s = u_s \vec{c}$ for some $u_s \in \mathbf{R}$. Equation (3.4) is solvable for

$$(m+1)k < s < (m+2)k$$

if and only if

$$\pi \sum_{l=1}^{s-1} \widehat{U}_l Q \vec{u}_{s-l} + \pi \sum_{l=0}^{s-n} \widehat{U}_l Q \vec{u}_{s-l-n} = 0.$$

But

$$\pi \sum_{l=0}^{s-n} \widehat{U}_l Q \vec{u}_{s-l-n} = 0.$$

So

$$\pi \sum_{l=1}^{s-1} \widehat{U}_l Q \vec{u}_{s-l} = u_{s-k}(\pi \widehat{U}_k Q \vec{c} + \pi \widehat{C} Q \vec{u}_k) + \pi \sum_{l=1}^{s-1} \widehat{U}_l Q \vec{u}_{s-l} = 0.$$

From this equation one can get subsequently that $u_{mk+1}, \ldots, u_{mk+k-1} = 0$. □

By virtue of this lemma we obtain

$$u_a(z) = \sum_{s \geq 0} u_{as}(1-z)^{s/2}.$$

We will show that equations (3.3), (3.4) uniquely define u_{as}.

As we know, $\vec{u}_1 = u_1 \vec{c}$. From (3.3) we have

$$\pi(\widehat{U}_1 Q \vec{u}_1 - Q\vec{1}) = 0.$$

Hence

$$u_1 = \pm \frac{\pi Q \vec{1}}{\pi \widehat{C} Q \vec{c}},$$

which corresponds to different branches of the function $u_a(z)$. We should choose "$-$", because $u_a(z)$ is increasing in a neighbourhood of 1. Assume that we have determined $\vec{u}_s$, $s < m$, then we can write $\vec{u}_s$ as $\vec{u}_s = u_s\vec{c} + \vec{v}_s$, where $\vec{v}_s \perp \pi$. For $s > 1$, equation (3.4) is solvable if and only if

$$\Big(\sum_{l=1}^{s-1} \widehat{U}_l Q \vec{u}_{s-l} + \pi \sum_{l=0}^{s-2} \widehat{U}_l Q \vec{u}_{s-l-2}\Big) \perp \pi. \tag{3.6}$$

Hence by (3.4)

$$\vec{v}_m = (\widehat{Q} - Q)^{-1} \Big(\sum_{l=1}^{m-1} \widehat{U}_l Q \vec{u}_{s-l} + \sum_{l=0}^{m-2} \widehat{U}_l Q \vec{u}_{s-l-2}\Big).$$

By plugging $s = m + 1$ into (3.6), we get the relation

$$\pi \widehat{U}_m Q \vec{u}_1 + \pi \widehat{U}_1 Q \vec{u}_m = \pi \sum_{l=1}^{m-1} \widehat{U}_l Q \vec{u}_{m+1-l} + \pi \sum_{l=0}^{m-1} \widehat{U}_l Q \vec{u}_{m-l-1},$$

thus yielding u_m. The theorem is proved. □

Now all ingredients for the analytical proof of Theorem 1.4 are available.

Let $z \in \mathbf{C}, |z| \le 1$, $\vec{z} = (z_1, \ldots, z_r) \in \mathbf{C}^r, a \in R, |z_a| \le 1$.

Denote by $F(z, \vec{z})$ the doubly generating function

$$F(z, \vec{z}) = \sum_t z^t \mathsf{E}_* \prod_a z_a^{n_a(t)},$$

where

$$\mathsf{E}_* \prod_a z_a^{n_a(t)} = \sum_n \sum_{a_1,\ldots,a_n} p_t(a_1 \ldots a_n) z_{a_1} \ldots z_{a_n}.$$

By Lemma 2.1 we have

$$F(z, \vec{z}) = \sum_n \sum_{a_1,\ldots,a_n} z^n \varphi_*(z) q_{*a_1} \varphi_{a_1}(z) q_{a_1 a_2} \varphi_{a_2}(z) \ldots q_{a_{n-1} a_n} \varphi_{a_n}(z) z_{a_1} \ldots z_{a_n}, \tag{3.7}$$

or in matrix notation

$$F(z, \vec{z}) = \varphi_*(z) \Big(1 + \sum_{a,b} q_{*a} z z_a (I - H(z, \vec{z}))^{-1}_{ab}\Big),$$

where

$$H(z, \vec{z})_{ab} = z z_b q_{ab} \varphi_b(z) = z z_b q_{ab} u_b(z)/q_b.$$

So we can write $F(z, \vec{z})$ in the form

$$F(z, \vec{z}) = \frac{T(z, \vec{z})}{(1 - z^2 \sum_a q_{*a} u_a(z^2)) \det(I - H(z, \vec{z}))}, \tag{3.8}$$

where $T(z, \vec{z})$ is some polynomial in $z, z_1, \ldots, z_r$.

We will use the following notation: if $f(z, \vec{z})$ is some function of $z, \vec{z}$, then for any $y \in \mathbf{C}$, we define $f(z, y) = f(z, \vec{z}) \mid_{\vec{z}=(y,\ldots,y)}$. We have

$$\begin{aligned}\lim_{t\to\infty} \mathsf{E}_* e^{-\frac{n(t)x}{\sqrt{t}}} &= \lim_{t\to\infty} \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{1}{z^{t+1}} F(z, e^{-\frac{x}{\sqrt{t}}}) dz \\ &= \lim_{t\to\infty} \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{1}{z^{t+1}} \frac{T(z, e^{-\frac{x}{\sqrt{t}}})}{(1 - z^2 \sum_a q_{*a} u_a(z^2)) \det(I - H(z, e^{-\frac{x}{\sqrt{t}}}))} dz.\end{aligned}$$

This integral can be split into two parts: one over

$$L = \{z \in \mathbf{C}, |z| = 1, |\arg z| < \varepsilon\}$$

and the other one over $\{z \in \mathbf{C}, |z| = 1\} \backslash L$. By the Riemann – Lebesgue theorem, the integral over $\{z \in \mathbf{C}, |z| = 1\} \backslash L$ tends to 0 as $t \to \infty$, because we can write it in the form

$$\int\limits_{[-\pi,\pi]\backslash(-\varepsilon,\varepsilon)} e^{-ity} (f(y) + O(1/\sqrt{t})) dy,$$

where $f(y) \in L^1([-\pi, \pi] \backslash (-\varepsilon, \varepsilon))$.

Therefore, it is sufficient to consider only the integral over L. Define $s = \sqrt{1-z}$ and let

$$\begin{aligned}\tilde{T}(s, e^{-\frac{x}{\sqrt{t}}}) &= T(z, e^{-\frac{x}{\sqrt{t}}}), \\ u(s) &= \frac{\sqrt{1-z}}{(1 - z^2 \sum_a q_{*a} u_a(z^2))}, \\ h(s, e^{-\frac{x}{\sqrt{t}}}) &= \det(I - H(z, e^{-\frac{x}{\sqrt{t}}})).\end{aligned}$$

The function $\tilde{T}(s, y)$ is holomorphic in s in a neighbourhood of $s = 0$ and it is a polynomial in y. The function $u(s)$ is holomorphic in a neighbourhood of 0. Moreover,

$$\begin{aligned}h(0, 1) &= 0, \\ \frac{\partial h}{\partial s}(0, 1) &\neq 0.\end{aligned}$$

So there is a unique function f of the variable $1-y$ solving the equation $h(f, y) = 0$ in a neighbourhood of $(0, 1)$.

We want to use s as the integration variable for the above integral. Hence, we integrate over the path $s(L)$. But instead of this path, we can consider the simpler one $[-\varepsilon i, \varepsilon i]$. This is because the integral over $L' \cup L''$ in Figure 3.1 tends

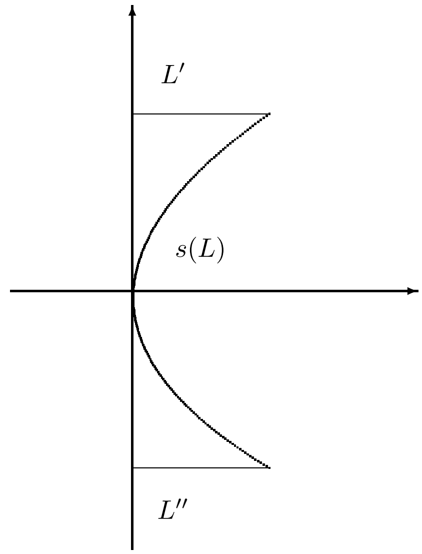

Figure 3.1

to 0, since $|z| \geq 1$ on $L' \cup L''$. So it is sufficient to calculate

$$-\lim_{t\to\infty} \frac{1}{\pi i} \int_{-\varepsilon i}^{\varepsilon i} \frac{1}{(1-s^2)^t} \frac{u(s)\tilde{T}(s, e^{-\frac{x}{\sqrt{t}}})}{h(s, e^{-\frac{x}{\sqrt{t}}})} ds.$$

Define the function

$$\psi(s,y) = \frac{u(s)\tilde{T}(s,y)}{h(s,y)} - \frac{u(f(1-y))\tilde{T}(f(1-y),y)}{(s-f(1-y))\frac{\partial}{\partial s}h(f(1-y),y)},$$

for s in a neighbourhood of 0 and $y \in (1-\varepsilon', 1+\varepsilon')$. Because $\psi(\cdot, y)$ is holomorphic in a neighbourhood of 0,

$$\Big| \int_{-\varepsilon i}^{\varepsilon i} \frac{1}{(1-s^2)^t} \psi(s,y) \Big| \leq C\varepsilon.$$

Since ε can be arbitrary small, we can neglect this integral. So it is sufficient to consider

$$\lim_{t\to\infty} \frac{u(f(1-e^{-\frac{x}{\sqrt{t}}}))\tilde{T}(f(1-e^{-\frac{x}{\sqrt{t}}}), e^{-\frac{x}{\sqrt{t}}})}{\frac{\partial}{\partial s}h(f(1-e^{-\frac{x}{\sqrt{t}}}), e^{-\frac{x}{\sqrt{t}}})} \Big(-\frac{1}{\pi i} \int_{-\varepsilon i}^{\varepsilon i} \frac{(1-s^2)^{-t}}{s-f(1-e^{-\frac{x}{\sqrt{t}}})} ds \Big)$$

$$= \frac{u(0)\tilde{T}(0,1)}{\frac{\partial}{\partial s}h(0,1)} \lim_{t\to\infty} \frac{1}{\pi i} \int_{-\varepsilon i}^{\varepsilon i} \frac{(1-s^2)^{-t}}{f(1-e^{-\frac{x}{\sqrt{t}}})-s} ds.$$

Hence we should compute

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{\pi i} \int_{-\varepsilon i}^{\varepsilon i} \frac{(1-s^2)^{-t}}{f(1-e^{-\frac{x}{\sqrt{t}}})-s} ds \\
&= \lim_{t\to\infty} \frac{2}{\pi} \int_0^{\varepsilon} (1+s^2)^{-t} \Big(\frac{1}{f(1-e^{-\frac{x}{\sqrt{t}}})-si} + \frac{1}{f(1-e^{-\frac{x}{\sqrt{t}}})+si} \Big) ds \\
&= \lim_{t\to\infty} \frac{4}{\pi} \int_0^{\varepsilon} (1+s^2)^{-t} \frac{f(1-e^{-\frac{x}{\sqrt{t}}})}{f^2(1-e^{-\frac{x}{\sqrt{t}}})+s^2} ds \\
&= \lim_{t\to\infty} \frac{4}{\pi} \int_0^{\varepsilon/f(1-e^{-\frac{x}{\sqrt{t}}})} (1+s^2 f^2(1-e^{-\frac{x}{\sqrt{t}}}))^{-t} \frac{1}{1+s^2} ds \\
&= \lim_{t\to\infty} \frac{4}{\pi} \int_0^{\pi/2} \mathbf{1}_{[0,\arctan \varepsilon/f(1-e^{-\frac{x}{\sqrt{t}}})]}(\theta)(1+\tan^2\theta f^2(1-e^{-\frac{x}{\sqrt{t}}}))^{-t} d\theta.
\end{aligned}$$

For any $\theta \in [0,\pi)$,

$$0 < \big(1+\tan^2\theta f^2(1-e^{-\frac{x}{\sqrt{t}}})\big)^{-t} \le 1,$$

and

$$\exp\big\{-\tan^2\theta \lim_{t\to\infty} t f^2\big(1-e^{-\frac{x}{\sqrt{t}}}\big)\big\} = \exp\{-\tan^2\theta[xf'(0)]^2\}.$$

By virtue of Lebesgue's theorem, the last expression equals

$$\begin{aligned}
\frac{2}{\pi}\int_0^{\pi/2} e^{-\tan^2\theta[xf'(0)]^2} d\theta &= \frac{2}{\pi}\int_0^{\infty} \frac{e^{-z^2[xf'(0)]^2}}{1+z^2} dz \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-z^2[xf'(0)]^2}}{1+z^2} dz.
\end{aligned}$$

The function

$$\frac{e^{-z^2[xf'(0)]^2}}{1+z^2}$$

is the characteristic function of the sum of two independent random variables $w+\eta$, where w is normally $N(0,2[xf'(0)]^2)$ distributed and η has the Laplace distribution with density $e^{-|z|}$. By the inverse Fourier transformation

$$\int_{-\infty}^{\infty} e^{-itz} \frac{e^{-z^2[xf'(0)]^2}}{1+z^2} dz = \int_{-\infty}^{\infty} e^{-|y-t|} \frac{1}{\sqrt{2\pi 2[xf'(0)]^2}} e^{-\frac{y^2}{4[xf'(0)]^2}} dy.$$

Putting $t = 0$ and $y = xv$ we get

$$\int_{-\infty}^{\infty} \frac{e^{-z^2[xf'(0)]^2}}{1+z^2} dz = \int_{-\infty}^{\infty} e^{-x|v|} \frac{1}{\sqrt{2\pi 2[f'(0)]^2}} e^{-\frac{v^2}{4[f'(0)]^2}} dv.$$

For any $x \in \mathbf{R}$, we then get

$$\lim_{t\to\infty} \mathsf{E}_* e^{-x\frac{n(t)}{\sqrt{t}}} = \mathsf{E} e^{x|w|} = \int_{-\infty}^{\infty} e^{-x|v|} \frac{1}{\sqrt{2\pi 2[f'(0)]^2}} e^{-\frac{v^2}{4[f'(0)]^2}} dv,$$

where w is normally $N(0, 2[f'(0)]^2)$ distributed. $\square$

A stronger version of the central limit theorem can be proved similarly.

Proof of Theorem 1.5. From (3.7) we have

$$\lim_{t\to\infty} \mathsf{E}_* e^{-\frac{1}{\sqrt{t}}\langle \vec{x}, \vec{n}(t)\rangle} = \lim_{t\to\infty} \frac{1}{2\pi i} \int_{|z|=1} \frac{F(z, \exp(-\vec{x}/\sqrt{t}))}{z^{t+1}} dz,$$

where $\vec{x} = (x_1, \dots, x_r) \in \mathbf{C}^r$, $\exp(-\vec{x}/\sqrt{t}) = (e^{-\frac{x_1}{\sqrt{t}}}, \dots, e^{-\frac{x_r}{\sqrt{t}}})$. By (3.8) it is equal to

$$\lim_{t\to\infty} \frac{1}{2\pi i} \int_{|z|=1} \frac{T(z, \exp(-\vec{x}/\sqrt{t}))}{z^{t+1}(1 - z^2 \sum_a q_{*a} u_a(z^2)) \det(I - H(z, \exp(-\vec{x}/\sqrt{t})))}.$$

The function $\det(I - H(z, \vec{z}))$ is a polynomial in $(z_1, \dots, z_r)$ and it is holomorphic on z in $\{z \in \mathbf{C} : |z| < 1\} \cup V_\varepsilon$, where V_ε is a neighbourhood of 1 not containing the segment $[1, 1+\varepsilon]$, i.e. $V_\varepsilon = \{z \in \mathbf{C} : |z-1| < \varepsilon\} \backslash [1, 1+\varepsilon]$.

As in the above, we define $s = \sqrt{1-z}$ for $z \in V_\varepsilon$. Then $h(s, \vec{z}) = \det(I - H(z(s), \vec{z}))$ is holomorphic in s near 0 and holomorphic in $s, z_1, \dots, z_r$ near $(0, 1, \dots, 1)$. As we know,

$$h(0, \vec{1}) = 0,$$
$$\frac{\partial h}{\partial s}(0, \vec{1}) \neq 0.$$

So there is a unique function $f(\vec{z})$ satisfying the equation $h(s, \vec{z}) = 0$ near $(0, \vec{1})$. The same computation as in the above can be used to obtain

$$\frac{1}{\sqrt{t}} \langle \vec{x}, \vec{n}(t) \rangle \to |w_{\vec{x}}|,$$

where $w_{\vec{x}}$ normally $N\Big(0, 2\Big[\frac{df}{d\vec{x}}(\vec{1})\Big]^2\Big)$ distributed. Indeed, write

$$\frac{df}{d\vec{x}}(\vec{1}) = \sum_{a\in R} x_a \frac{\partial}{\partial z_a} h(\vec{1}) x_a = \frac{1}{\frac{\partial}{\partial s} h(0,\vec{1})} \sum_{a\in R} x_a \frac{\partial}{\partial z_a} h(0,\vec{1}).$$

From the definition of $H(z,\vec{z})$ and Theorem 3.2 we have

$$\begin{aligned} \frac{\partial}{\partial s} h(0,\vec{1}) &= \sum_{a\in R} u_{a1} m_{aa}, \\ \frac{\partial}{\partial z_a} h(0,\vec{1}) &= m_{aa}, \end{aligned}$$

where m_{aa} is the minor of the matrix $(I - H(1,\vec{1}))$ corresponding to the (a,a)-th element. So,

$$\frac{df}{d\vec{x}}(\vec{1}) = \frac{\sum_{a\in R} x_a m_{aa}}{\sum_{a\in R} u_{a1} m_{aa}}.$$

Hence $w_{\vec{x}} = \langle \vec{x}, \vec{w} \rangle$, where $\vec{w}$ is normally distributed with covariance matrix $C = \{c_{ab}\}_{a,b\in R}$, given by

$$c_{ab} = 2 \frac{m_{aa} m_{bb}}{\big(\sum_{a\in R} u_{a1} m_{aa}\big)^2}.$$

In other words, $\vec{w} = w_1 \vec{c}$ with $\vec{c} \in \mathbf{R}^r$. □

4. Appendix

Here we a consider general "inhomogeneous" renewal equation.

Let $f = \{f_n\}_{n\geq 1}$ be a fixed probability distribution on $\mathbf{N}$, i.e.

$$\begin{aligned} f_n &\geq 0 \text{ for all } n \geq 1, \\ \sum_{n\geq 1} f_n &= 1. \end{aligned}$$

Define $\mathrm{supp}(f) = \{n : f_n > 0\}$. For each $n \in \mathrm{supp}(f)$, define $d_n = \min\{d > 0 : f_{n+d} > 0\}$ and

$$d(f) = \sup_{n\in \mathrm{supp}(f)} d_n.$$

Let $F = \{f^k\}_{k\geq 1}$ be a sequence of probability distributions on $\mathbf{N}$. So for $f^k = \{f^k_n\}_{n\geq 1}$ we have

$$\begin{aligned} f^k_n &\geq 0 \text{ for all } n \geq 1, \\ \sum_{n\geq 1} f^k_n &= 1. \end{aligned}$$

We will call F a renewal distribution. Let $c_1 > 0$, $c_2 > 1$ be fixed. Denote by C the set of all renewal distributions $F = \{f^k\}_{k\geq 1}$ such that for all $k, n \geq 1$

$$c_1 f_n \leq f_n^k \leq c_2 f_n. \tag{4.1}$$

Let $\{\tau_k\}_{k\geq 1}$ be independent random variables with values in $\mathbf{N}$ "having renewal distribution" $F = \{f^k\}_{k\geq 1}$, i.e.

$$\mathsf{P}\{\tau_k = n\} = f_n^k.$$

For $n \geq 1$ define

$$p_n(F) = \mathsf{P}\{\tau_1 + \cdots + \tau_k = n, \text{ for some } k \geq 1\}.$$

Theorem 4.1. *Assume that*

$$\begin{aligned} d(f) &< \infty, \\ \sum_n n f_n &= \infty. \end{aligned}$$

Then there exists $\psi_n \to 0$ as $n \to \infty$, such that for all $F \in C$

$$p_n(F) \leq \psi_n.$$

In other words, $p_n(F)$ tends to 0 uniformly on C.

Proof. Let $a = \{a_n\}_{n\geq 0}$, $b = \{b_n\}_{n\geq 0}$ be two sequences. By $(a, b) = \{(a, b)_n\}_{n\geq 1}$ we denote the convolution of a, b, i.e.

$$(a, b)_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Let $F = \{f^k\}_{k\geq 1} \in C$. It will be convenient to define $f_0^k = 0$ for all $k \geq 1$. By the definition of $p_n(F)$ we can write

$$p_n(F) = \sum_{k\geq 1} (f^1, \ldots, f^k)_n = f_n^1 + \sum_{k\geq 1} \sum_{t=0}^{n} (f^1, \ldots, f^k)_{n-t} f_t^{k+1}. \tag{4.2}$$

For $k \geq 1$ define $r^k = \{r_n^k\}_{n\geq -1}$ with

$$r_n^k = \sum_{l\geq n+1} f_l^k.$$

Define $r = \{r_n\}_{n\geq -1}$ similarly, but using the distribution f. Remark that

$$\sum_{n\geq 1} nf = \sum_{k\geq 0} r_k = \sum_{k\in \text{supp}(f)} d_k r_k \leq d(f) \sum_{k\in \text{supp}(f)} r_k.$$

Hence

$$\sum_{k\in\mathrm{supp}(f)} r_k = \infty. \tag{4.3}$$

For all $k \geq 1$, $\tau_k > 0$. Hence for any $n \geq 0$

$$\begin{aligned} 1 &= \mathsf{P}\{\tau_1 + \cdots + \tau_k > n, \text{ for some } k \geq 1\} \\ &= \mathsf{P}\{\tau_1 + \cdots + \tau_{n+1} > n\} \\ &= \mathsf{P}\{\tau_1 > n\} + \sum_{k=1}^{n} \mathsf{P}\Big\{\sum_{i=1}^{k} \tau_i \leq n, \sum_{i=1}^{k+1} \tau_i > n\Big\} \\ &= r_n^1 + \sum_{k=1}^{n}\sum_{t=0}^{n} \mathsf{P}\Big\{\sum_{i=1}^{k} \tau_i = t\Big\} \mathsf{P}\{\tau_{k+1} > n-t\} \\ &= r_n^1 + \sum_{k\geq 1}\sum_{t=0}^{n} (f^1, \ldots, f^k)_{n-t} r_{n-t}^{k+1}. \end{aligned}$$

Condition (4.1) implies that $r_t \leq r_t^{k+1}/c_1$. So we get for any $F \in C$ that

$$\sum_t p_{n-t}(F) r_t \leq \frac{1}{c_1}. \tag{4.4}$$

Next we will show that $p_n(F)$ tends to 0 uniformly on C. The idea is the same as in homogeneous case (see for example Chapter 1.6 of [11]). We show that the convergence not being uniform on C contradicts (4.4). To this end, define the shift operator Θ on renewal distributions F as follows: if $F = \{f^k\}_{k\geq 1} \in C$ then $\Theta F = \{f^{k+1}\}_{k\geq 1}$. It is clear, that $\Theta F \in C$ and that

$$p_n(F) = \sum_{k=0}^{n} f_k^1 p_{n-k}(\Theta F).$$

Let

$$\begin{aligned} \lambda_n(F) &= \sup_{k\geq n} p_k(F), \\ \lambda_n &= \sup_{F\in C} \lambda_n(F). \end{aligned}$$

It is clear that $\lambda_{n+1}(F) \leq \lambda_n(F)$, for any F, and hence $\lambda_{n+1} \leq \lambda_n$. So the sequence λ_n has a limit,

$$\lim_{n\to\infty} \lambda_n = \lambda.$$

We should prove that $\lambda = 0$. Assume that $\lambda \neq 0$. We can choose a sequence $F_n = \{f^m(n), m \in \mathbf{N}\} \in C$, $n \geq 1$ such that

$$\lim_{n\to\infty} \lambda_n(F_n) = \lambda.$$

Let us choose a sequence $\{m_n\}_{n\geq 0}$, such that

$$p_{m_n}(F_n) \to \lambda, \text{ as } n \to \infty.$$

By definition

$$p_{m_n}(F_n) = \sum_{k=0}^{m_n} f_k^1(n) p_{m_n-k}(\Theta F_n).$$

Remark that for any $k \geq 0$,

$$\limsup_{n\to\infty} p_{m_n-k}(\Theta F_n) \leq \lim_{n\to\infty} \lambda_{m_n-k} = \lambda.$$

Let us show that for $k \in \mathrm{supp}(f)$

$$\liminf_{n\to\infty} p_{m_n-k}(\Theta F_n) = \lambda. \tag{4.5}$$

For $\varepsilon > 0$, there exists N such that

$$\sum_{n\geq N} f_n < \varepsilon.$$

From (4.1) we have for any $n \geq 1$ that

$$\sum_{k=N}^{m_n} f_k^1(n) p_{m_n-k}(\Theta F_n) \leq c_2 \varepsilon$$

and

$$p_{m_n}(F_n) - c_2\varepsilon \leq \sum_{k=0}^{N} f_k^1(n) p_{m_n-k}(\Theta F_n).$$

Let $k_0 \leq N, k_0 \in \mathrm{supp}(f)$. Then

$$\begin{aligned} p_{m_n}(F_n) - c_2\varepsilon \quad &\leq \quad f_{k_0}^1(n) p_{m_n-k_0}(\Theta F_n) \\ &\qquad + \sum_{k=0,k\neq k_0}^{N} f_k^1(n) p_{m_n-k}(\Theta F_n). \end{aligned}$$

If

$$\lambda_{\inf}(k) = \liminf_{n\to\infty} p_{m_n-k}(\Theta F_n),$$

then

$$\begin{aligned} &\liminf_{n\to\infty} \Big(f_{k_0}^1(n) p_{m_n-k_0}(\Theta F_n) + \sum_{k=0,k\neq k_0}^{N} f_k^1(n) p_{m_n-k}(\Theta F_n) \Big) \\ &\leq \quad \limsup_{n\to\infty} \Big(f_{k_0}^1(n) \lambda_{\inf}(k_0) + \sum_{k=0,k\neq k_0}^{N} f_k^1(n) \lambda \Big) \end{aligned}$$

$$\leq \limsup_{n\to\infty} \left(f^1_{k_0}(n)\lambda_{\inf}(k_0) + \lambda(1 - f^1_{k_0}(n))\right)$$
$$\leq \lambda + (\lambda_{\inf}(k_0) - \lambda)c_1 f_{k_0}.$$

We get

$$\lambda - c_2\varepsilon = \liminf_{n\to\infty} p_{m_n}(F_n) - c_2\varepsilon \leq \lambda + (\lambda_{\inf}(k_0) - \lambda)c_1 f_{k_0}.$$

The constant ε can be chosen arbitrarily small in this inequality. Since the right-hand side does not depend on ε and since $f_{k_0} > 0$, this implies

$$\lambda \leq \lambda_{\inf}(k_0).$$

Consequently (4.5) holds. From (4.4) we have

$$\sum_{k\in \mathrm{supp}(f)\cap[0,m_n]} p_{m_n-k}(\Theta F_n)r_k \leq \sum_{k=0}^{m_n} p_{m_n-k}(\Theta F_n)r_k \leq \frac{1}{c_1}. \tag{4.6}$$

If $\lambda \neq 0$, then (4.5) implies that

$$\sum_{k\in \mathrm{supp}(f)\cap[0,m_n]} p_{m_n-k}(\Theta F_n)r_k \to \infty, \quad \text{as } n \to \infty,$$

thus contradicting (4.6). Hence, $\lambda = 0$ and so the theorem is proved. □

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